Math 170B Midterm 1

May 3, 2019

- Please put your name, ID number, seat number, and sign and date.
- There are 4 problems worth a total of 100 points.
- You must show your work to receive credit.

Print Name: ________________________________________________

Student ID: ________________________________________________

Seat Number: __________

Signature and Date: _________________________________________

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Formulas and Definitions:

- $f \in C^k[a,b]$ means $f, f', \ldots, f^{(k)}$ all exist and are continuous in $[a,b]$.
- $f \in C^k(\mathbb{R})$ means $f, f', \ldots, f^{(k)}$ all exist and are continuous for all real numbers.
- Intermediate Value Theorem: if $f \in C[a,b]$, for $a < b$, and $f(a)$ and $f(b)$ have different signs, then there exists a root of $f$ in $[a,b]$.
- Mean Value Theorem: if $f \in C^1[a,b]$ and $x, y \in [a,b]$, then $f(x) - f(y) = f'(\xi)(x - y)$, for some $\xi$ between $x, y$.
- Taylor series: if $f \in C^{n+1}[a,b]$, and $x, y \in [a,b]$, then
  $$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(y)}{k!} (x - y)^k = \sum_{k=0}^{n} \frac{f^{(k)}(y)}{k!} (x - y)^k + \frac{f^{(n+1)}(\xi(x))}{(n+1)!} (x - y)^{n+1},$$
  for some $\xi(x)$ between $x, y$.
- Bisection method starts with interval $[a_0, b_0]$ and uses: $c_n = \frac{a_n + b_n}{2}$.
- Fixed point iterations: $x_{n+1} = F(x_n)$.
- $F$ is a contractive map in $[a, b]$ if there exists $0 \leq \lambda < 1$ such that $|F(x) - F(y)| \leq \lambda|x - y|$, for all $x, y \in [a, b]$.
- Newton’s method: $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$.
- Newton’s method for system of equations, $\vec{f}(\vec{x}) = 0$, is $\vec{x}^{(n+1)} = \vec{x}^{(n)} - J^{-1}(\vec{x}^{(n)}) \vec{f}(\vec{x}^{(n)})$, where $J(\vec{x})$ is the Jacobian matrix with $J_{ij}(\vec{x}) = \frac{\partial f_i(\vec{x})}{\partial x_j}$.
- $[a \ b]^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
- Secant method: $x_{n+1} = x_n - \frac{f(x_n)(x_n - x_{n-1})}{f(x_n) - f(x_{n-1})}$.
- Method of false position is a variation of bisection method where, for each interval $[a_n, b_n]$, approximations and interval cut locations are both chosen at $b_n - \frac{f(b_n)(b_n - a_n)}{f(b_n) - f(a_n)}$.
- If $\{x_n\}_{n=0}^{\infty}$ converges to $r$, then it has order of convergence $q > 0$ if $\lim_{n \to \infty} \frac{|x_{n+1} - r|}{|x_n - r|^q}$ exists and is nonzero.

Do not remove this page. You can use below and back for scratch work.
1. (25 pts) Let $f$ be a continuous function and suppose we already have the Matlab function “f.m”, with header “function [value] = f(x)”, that returns values of $f(x)$. Given the following header for a Matlab function:

$$\text{function [approx] = FalsePosition(a,b,N)}$$

that inputs

- starting interval $[a, b]$, where $a < b$ and $f(a)$ and $f(b)$ have opposite signs;
- approximation number $N \geq 0$;

complete this function so that it outputs $c_N$ of the method of false position for finding a root of $f$. Make sure your program uses at most $N + 3$ calls to “f.m”.

```matlab
function [approx] = FalsePosition(a,b,N)
    fa = f(a);
    fb = f(b);
    approx = b-fb*(b-a)/(fb-fa);
    for k = 1:N
        fc = f(approx);
        if fa*fc <= 0
            b = c;
            fb = fc;
        else
            a = c;
            fa = fc;
        end
        approx = b-fb*(b-a)/(fb-fa);
    end
end
```
2. (25 pts) **Compute** the approximations in the following problems:

(a) Consider

\[4x^3 + 8t^2 - 2x + 2t - 3/2 = 0,\]

which defines \(x\) implicitly as a function of \(t\): \(x = x(t)\). Find **bisection method**’s \(c_2\) approximation of \(x(1/4)\), using starting interval \([0, 1]\).

(b) Given

\[
\begin{align*}
  x_1^2 + 2x_1x_2 + 1 &= 0 \\
  x_1^2x_2 + 2x_1 &= 0,
\end{align*}
\]

perform **one** iteration of **Newton’s method** with initial guesses \(x_1^{(0)} = 1, x_2^{(0)} = -3\).

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(a) \(t = 1/4\) implies

\[0 = 4x^3 + 1/2 - 2x + 1/2 - 3/2 = 4x^3 - 2x - 1/2,
\]

so let \(f(x) = 4x^3 - 2x - 1/2\).

\[
\begin{array}{ccc|ccc}
  a & c & b & f(a) & f(c) & f(b) \\
  0 & 1/2 & 1 & -1/2 & -1 & 3/2 \\
  1/2 & 3/4 & 1 & -1 & -5/16 & 3/2 \\
  3/4 & 7/8 & 1 & -5/16 & 3/2 \\
\end{array}
\]

So \(c_2 = 7/8\).

(b) Let \(f_1(x_1, x_2) = x_1^2 + 2x_1x_2 + 1\) and \(f_2(x_1, x_2) = x_1^2x_2 + 2x_1\). The Jacobian matrix

\[
J(x_1, x_2) = \begin{bmatrix}
2x_1 + 2x_2 & 2x_1 \\
2x_1x_2 + 2 & x_1^2
\end{bmatrix},
\]

and

\[
J(1, -3) = \begin{bmatrix}
-4 & 2 \\
-4 & 1
\end{bmatrix},
\]

so

\[
J^{-1}(1, -3) = \frac{1}{4} \begin{bmatrix}
1 & -2 \\
4 & -4
\end{bmatrix}.
\]

Thus

\[
\begin{bmatrix}
x_1^{(1)} \\
x_2^{(1)}
\end{bmatrix} = \begin{bmatrix}
1 \\
-3
\end{bmatrix} - \frac{1}{4} \begin{bmatrix}
1 & -2 \\
4 & -4
\end{bmatrix} \begin{bmatrix}
-4 \\
-1
\end{bmatrix} = \begin{bmatrix}
3/2 \\
0
\end{bmatrix}.
\]
3. (25 pts) **Answer** the following questions:

(a) Let \( f \in C[-2, 3] \) and suppose \( f(-2) \) and \( f(3) \) have different signs. Find the first \( n \) such that the absolute error of \( c_n \), computed by **bisection method**, is guaranteed by **error bounds** to be \( \leq 10^{-4} \). You may leave your answer unsimplified.

(b) Let \( f(x) = \frac{x(x^2+1)}{3} - x^2 + 1 \). Determine whether or not \( f \) is a **contractive map** in \([0, 3/2]\).

(a) Error bound is

\[
|r - c_n| \leq \frac{3 - (-2)}{2^{n+1}} = \frac{5}{2^{n+1}}.
\]

So

\[
\frac{5}{2^{n+1}} \leq 10^{-4}
\]

ensures absolute error of \( c_n \) is \( \leq 10^{-4} \). We solve for \( n \):

\[
n \geq \log_2(5 \cdot 10^4) - 1.
\]

(b) Since \( f \in C^1[0, 3/2] \), we can study

\[
f'(x) = x^2 + 1/3 - 2x,
\]

and see if \(|f'(x)| \leq \lambda < 1\), for some \( \lambda \geq 0 \). We do this by using calculus to find \( \max_{x \in [0,3/2]} |f'(x)| \). Critical points of \( f' \) are at

\[
0 = f''(x) = 2x - 2,
\]

so \( x = 1 \in [0,3/2] \). Checking \(|f'(x)|\) at critical points and endpoints, we see \(|f'(1)| = 2/3, |f'(0)| = 1/3, |f'(3/2)| = 5/12\), so \( \max_{x \in [0,3/2]} |f'(x)| = 2/3 \). Thus, Mean Value Theorem says for any \( x, y \in [0,3/2] \), there exists \( \xi \in [0,3/2] \) and

\[
|f(x) - f(y)| = |f'(\xi)||x - y| \leq \lambda |x - y|,
\]

for \( \lambda = 2/3 < 1 \). This means \( f \) is a contractive map.
4. (25 pts) Let \( f \in C^2(\mathbb{R}) \) and let \( \{x_n\}_{n=0}^{\infty} \) be a sequence of approximations generated by secant method. Suppose \( x_n \) converges to \( r \), a root of \( f \), and suppose \( f'(r) \neq 0 \). Prove, for \( n \) large, and \( e_n = x_n - r \),

\[
e_{n+1} \approx \frac{f''(r)}{2f'(r)} e_ne_{n-1}.
\]

See book, page 96, result (5).