Math 170B Midterm 2
March 2, 2018

- Please put your name, ID number, seat number, and sign and date.
- There are 4 problems worth a total of 100 points.
- You must show your work to receive credit.

Print Name: ________________________________

Student ID: ________________________________

Seat Number: __________

Signature and Date: __________________________

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Formulas and Definitions:

- $f \in C^k[a,b]$ means $f, f', \ldots, f^{(k)}$ all exist and are continuous in $[a,b]$
- A polynomial $P$ has degree $n$ means $P(x) = a_0 + a_1 x + \ldots + a_n x^n$, where $a_n \neq 0$
- Fundamental Theorem of Algebra: A nonzero polynomial of degree $n$ has exactly $n$ roots in the complex plane, counting multiplicity
- Generalized Rolle’s Theorem: if $f \in C^n[a,b]$ and $f(x) = 0$ at $n+1$ distinct numbers in $[a,b]$, then there exists $c \in (a,b)$ such that $f^{(n)}(c) = 0$.
- Given data points $(x_i, y_i)$, $i = 0, \ldots, n$, the nodes are defined to be the $x_i$, for $i = 0, \ldots, n$, and evenly spaced nodes have $x_{k+1} - x_k$ all the same value, for all $k = 0, \ldots, n-1$
- The Lagrange polynomial $P$ interpolating data points $(x_i, f(x_i))$, $i = 0, \ldots, n$, with distinct nodes, is the unique deg $\leq n$ polynomial satisfying $P(x_i) = f(x_i)$
  - Lagrange form:
    $$P(x) = \sum_{i=0}^{n} \left( f(x_i) \prod_{j=0, j \neq i}^{n} \frac{x-x_j}{x_i-x_j} \right)$$
  - Lagrange polynomial error formula: if $f \in C^{n+1}[a,b]$, and $(x_i, f(x_i))$, $i = 0, \ldots, n$, are data points with distinct nodes in $[a,b]$, and $P$ is the Lagrange polynomial interpolating the data points, then for all $x \in [a,b]$, there exists $\xi(x) \in (a,b)$ such that
    $$f(x) = P(x) + \frac{f^{(n+1)}(\xi(x))}{(n+1)!} \prod_{i=0}^{n} (x-x_i)$$
- For data points $(x_i, f(x_i))$, $i = 0, \ldots, n$, with nodes $x_0 < x_1 < \ldots < x_n$, the piecewise linear interpolant in $[x_i, x_{i+1}]$ is the line interpolating $(x_i, f(x_i)), (x_{i+1}, f(x_{i+1}))$
- Divided differences: zeroth divided difference is $f[x_i] = f(x_i)$ and $k$th divided difference is
  $$f[x_i, x_{i+1}, \ldots, x_{i+k}] = \frac{f[x_{i+1}, x_{i+2}, \ldots, x_{i+k}] - f[x_{i}, x_{i+2}, \ldots, x_{i+k-1}]}{x_{i+k} - x_i}$$
- Newton form:
  $$P(x) = f[x_0] + \sum_{k=1}^{n} \left( f[x_0, x_1, \ldots, x_k] \prod_{j=0}^{k-1} (x-x_j) \right)$$

Do not remove this page. You can use below and back for scratch work.
1. (25 pts) Write a **Matlab** function with the header

    function [value] = polynomialpiece(m,x,y,k,z)

that inputs:

- number of data points \( m \);
- vectors \( x \) and \( y \), both with \( m \) components, holding \( x \)- and \( y \)-coordinates, respectively, of data points, and where the components of \( x \) are **evenly spaced** and in increasing order;
- an even number \( k > 0 \), and a location \( z \in (x_{k/2}, x_{m+1-k/2}) \) distinct from the nodes;

and outputs, using **Lagrange form**, the value at \( z \) of the deg \( \leq k - 1 \) Lagrange polynomial that interpolates only the \( k \) data points with nodes **closest** to \( z \).
2. (25 pts) For each of the following problems, circle either True or False. You do not need to show your work.

(a) Given \( f \in C^\infty[-1, 1] \) and evenly spaced nodes \(-1 = x_0 < x_1 < \ldots < x_n = 1\), let \( P_n \) be the Lagrange polynomial interpolating the data points \((x_i, f(x_i))\). Then, for all \( x \in [-1, 1] \), we always have \( \lim_{n \to \infty} |f(x) - P_n(x)| = 0 \).

True or False

(b) Given any polynomial \( P \) of degree 4, and any 9 distinct nodes \( x_i, i = 0, \ldots, 8 \), then \( P \) has to be the Lagrange polynomial interpolating the data points \((x_i, P(x_i)), i = 0, \ldots, 8\).

True or False

(c) Given nodes \( x_0 < x_1 < \ldots < x_{100} \), and values \( y_i \) and \( w_i \) satisfying \( y_i = w_i \), for all \( i \neq 30 \), let \( P \) be the piecewise linear interpolant for data points \((x_i, y_i), i = 0, \ldots, 100\), and \( Q \) be the piecewise linear interpolant for data points \((x_i, w_i), i = 0, \ldots, 100\). Then, for all \( x \in [x_0, x_1] \), we always have \( P(x) = Q(x) \).

True or False

(d) Given nodes \(-1 = x_0 < x_1 < \ldots < x_n = 1\), let \( f \in C^2[-1, 1] \), with \( f''(x) < 0 \) for all \( x \in [-1, 1] \), and let \( P \) be the piecewise linear interpolant for data points \((x_i, f(x_i))\). Then, for all \( x \in [-1, 1] \), we always have \( f(x) \geq P(x) \).

True or False

(e) Given distinct nodes \( x_i \in [-1, 1], i = 0, \ldots, 5 \) and \( f \in C^\infty[-1, 1] \) such that \( P(x) = x^5 + 2x^4 + 3x^3 + 4x^2 + 5x + 6 \) is the Lagrange polynomial interpolating the data points \((x_i, f(x_i)), i = 0, \ldots, 5\). Then we always have \( f[x_0, x_1, \ldots, x_5] = 6 \).

True or False

(f) Given \((x_i, y_i), i = 0, \ldots, 8\) data points with distinct nodes, let \( P \) be the Lagrange polynomial interpolating these data points. If \( \deg P = 4 \), then, in the divided difference table for Newton form, all \( k \)th divided differences must be zero, for \( 5 \leq k \leq 8 \).

True or False

(g) \( P(x) = 2x^3 - x^2 + 2x + 4 \) is the Lagrange polynomial interpolating the data points \((-1, -1), (0, 4), (1, 9), (2, 20)\).

True or False

(h) Given data points \((x_i, y_i), i = 0, \ldots, n\), with nodes \( x_0 < x_1 < \ldots < x_n \), and satisfying \( y_i < 0 \), for \( i = 0, \ldots, n \), let \( P \) be the piecewise linear interpolant for the data points. Then, for all \( x \in [x_0, x_n] \), we always have \( P(x) < 0 \).

True or False
3. (25 pts) Let \((x_i, f(x_i)), i = 0, \ldots, 3\), be data points, where \(x_i = i + 2\), for \(i = 0, \ldots, 3\). Given the divided differences

\[
f[x_0] = 1, \quad f[x_0, x_1] = 2, \quad f[x_0, x_1, x_2] = -7, \quad f[x_0, x_1, x_2, x_3] = 9,
\]

add the data point \((0, 3)\), find a **Newton form** for the Lagrange polynomial interpolating all 5 data points.
4. (25 pts) Let \( f \in C^n[a,b] \) and let \( x_i \in [a,b], \ i = 0, \ldots, n, \) be distinct nodes. Prove there exists \( \xi \in (a,b) \) such that

\[
f[x_0, \ldots, x_n] = \frac{f^{(n)}(\xi)}{n!}.
\]