Math 174 Final
December 7, 2016

- Please put your name, ID number, and sign and date.
- There are 8 problems worth a total of 200 points.
- **You must show your work to receive credit.**

Print Name: ____________________________________________

Student ID: ____________________________________________

Signature and Date: ____________________________________

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<th>Problem</th>
<th>Score</th>
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<td>1</td>
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<td>2</td>
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Total: /200
1. (25 pts) Consider the ODE $x' = f(t, x)$, with $x(t_0) = x_0$. Suppose we have available for use a Matlab function with header

$$\text{function } [\text{value}] = f(t, x)$$

that evaluates the function $f(t, x)$. Now, given the following header for a Matlab function:

$$\text{function } [xN] = \text{predictorcorrector}(t0,x0,h,N)$$

complete this function so that it performs **predictor-corrector**, using **Midpoint method** for predictor and **Trapezoid method** for corrector, with inputs $t_0 = t0$, $x_0 = x0$, and stepsize $h$, to output the approximation $x_N$.

Use only basic programming, such as for loops and if statements, and do **not** use any of Matlab's vector-vector or matrix-vector operations. Remember: Midpoint method

$$x_{i+1} = x_i + hf\left(t_i + \frac{h}{2}, x_i + \frac{h}{2} f(t_i, x_i)\right)$$

and Trapezoid method

$$x_{i+1} = x_i + \frac{h}{2} (f(t_i, x_i) + f(t_{i+1}, x_{i+1})).$$
2. (25 pts) Let $g$ be continuously differentiable and suppose there exists a $0 < \lambda < 1$ such that $|g'(x)| \leq \lambda$ for all $x$ real numbers. Suppose $p$ is a fixed point of $g$, and let $p_{i+1} = g(p_i)$ give fixed point iterations for an initial guess $p_0$. Show $p_i$ converges to $p$. Do not use fixed point theorem on convergence, since you are being asked to prove it in a special case. Remember: Mean value theorem

$$f(x) - f(y) = f'(\xi)(x - y)$$

for some $\xi$ between $x$ and $y$. 
3. (25 pts) Use Taylor series to find \( p, k, \) and \( C \neq 0 \) such that:

\[
    f'(x) - \frac{-f(x + 2h) + 4f(x + h) - 3f(x)}{2h} = C h^p f^{(k)}(x) + O(h^{p+1}),
\]

for \( h \) small. Remember:

\[
    f(x + h) = f(x) + h f'(x) + \frac{h^2}{2!} f''(x) + \frac{h^3}{3!} f'''(x) + \ldots
\]
4. (25 pts) Consider the linear system $Ax = b$, with $A = D - L - U$ nonsingular, where $D$ is diagonal and nonsingular, $L$ is strictly lower triangular, and $U$ is strictly upper triangular. Consider the iterative method whose sequence of approximations satisfies:

$$x^{(k+1)} = (D - U)^{-1}Lx^{(k)} + (D - U)^{-1}b.$$ 

When

$$A = \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix},$$

write out the iteration matrix and use it to determine whether this iterative method’s sequence of approximations will converge to the solution of $Ax = b$ for any initial guess. Remember:

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}^{-1} = \frac{1}{a_{11}a_{22} - a_{12}a_{21}} \begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix}.$$
5. (25 pts) Solve the following short problems.

(a) Perform Gaussian elimination with **partial pivoting** with back substitution, as if in a **2-digit rounding** machine, to **solve** the linear system
\[
\begin{align*}
  x_1 + 2x_2 &= 2 \\
-3x_1 + 3x_2 &= 1.
\end{align*}
\]

(b) Let \( x_i, i = 0, \ldots, 20 \) be evenly spaced nodes from \( x_0 = -1 \) to \( x_{20} = 1 \). With \( y_i = 1 + x_i^2 \), let \( p(x) \) be the **piecewise linear** interpolant for the data points \((x_i, y_i), i = 0, \ldots, 20\). Find \( p(0.17) \).
6. (25 pts) Solve the following short problems.

(a) Find $a, B, D$ so that the following is a **free** or **natural** ($S''(\text{endpoints}) = 0$) cubic spline

$$S(x) = \begin{cases} 
2 - 5x - 3x^2 + ax^3, & \text{if } -1 \leq x < 0, \\
D - 5x + Bx^2 + x^3, & \text{if } 0 \leq x \leq 1
\end{cases}$$

(b) Consider the ODE system

$$\begin{align*}
y'(t) &= y(t) + z(t) + t \\
z'(t) &= ty(t) - z(t)
\end{align*}$$

with initial conditions $y(3) = y_0, z(3) = z_0$. Use Euler’s method with stepsize $h = 0.1$ and initial guesses $y_0 = 1, z_0 = 2$ to solve for $y_1$ and $z_1$. Remember Euler’s method:

$$x_{i+1} = x_i + hf(t, x_i).$$
7. (25 pts) Let \((x_i, y_i), i = 0, \ldots, n,\) be \(n + 1\) data points with distinct nodes. Let \(p(x)\) and \(q(x)\) be polynomials of degree \(\leq n\) that interpolate the data points. Prove \(p \equiv q.\)
8. (25 pts) Given the function $f(x)$, consider the data points $(a - 2h, f(a - 2h)), (a - h, f(a - h)), (a, f(a))$. Write down the Lagrange form of the interpolating polynomial for these data points. Then use it to approximate $f'(a)$, writing your result in the form

$$\frac{Af(a - 2h) + Bf(a - h) + Cf(a)}{h},$$

for constants $A, B, C$. 