Homework #1

1. Suppose we have a computer that performs 3-digit rounding. Find the absolute and relative errors for the following numbers computed in the computer:
   (a) 0.000224729
   (b) 42381900
   (c) 2012.34 + 3.114
   (d) 2.728 ∙ 0.0036
   (e) 0.3856121 – 0.3840256

2. Prove k-digit rounding of any real number $x$ leads to a floating point number $fl(x)$ with relative error $\leq 5 \cdot 10^{-k}$.

3. In a computer that performs 4-digit rounding, find examples of the following:
   (a) Positive real numbers $a, b$ such that $fl(fl(a) − fl(b))$ has absolute error $\leq 10^{-5}$ but relative error $\geq 90\%$.
   (b) Positive real numbers $a, b$ such that $fl(fl(a) − fl(b))$ has absolute error $\geq 10^{5}$ but relative error $\leq 10\%$.

4. Let $u = 5 \cdot 10^{-k}$, so the relative error of k-digit rounding of any real number is $\leq u$. We of course take $k \geq 1$, so note $u < 1$. Given positive, real numbers $x, y$, show $x + y$, performed in a $k$-digit rounding machine, has relative error $\leq 2u + O(u^2)$.

5. (a) Prove, using the Intermediate Value Theorem, that there is a root of $f(x) = x^2 − 3$ in the interval $[1, 2]$.
   (b) Starting with this interval, draw a description of the bisection method on $f(x)$, labeling the approximations $p_1, p_2, p_3$.
   (c) Compute the values of $p_1, p_2, p_3$.
   (d) Determine a bound on the absolute error of $p_4$ without computing $p_4$. Then compute $p_4$ and determine the actual absolute error (using the actual exact solution computed by calculator or computer). Does it satisfy the bound?

6. Suppose $f$ is continuous and $f(x) < 0$ in $[-1, 2]$, and $f(x) > 0$ in $[2.5, 4]$. Using the bisection method on $f(x)$ with starting interval $[-1, 4]$, compute the values of $p_1, p_2, p_3$.

7. Suppose $f$ is continuous in $[-3, 2]$ and $f(-3) < 0$ and $f(2) > 0$.
   (a) Let $p_{15}$ be bisection method’s approximation, applied to $f(x)$ with starting interval $[-3, 2]$, after 15 iterations. Bound the absolute error of this approximation without computing $p_{15}$.
   (b) Use error bounds to determine $n$ such that the absolute error of $p_n$ is guaranteed to be $\leq 10^{-12}$.
8. (Matlab)

(a) Using the “cos” command in Matlab, write a Matlab function that inputs a number \(x\) and outputs the value \(\cos x - x\). Print out or write out the function.

(b) Write a Matlab function that inputs the endpoints of a starting interval \(a, b\) and the number \(N\) and outputs the bisection method’s \(p_N\) approximation of the root for the function in part (a). Print out or write out the function.

(c) Run your program for the starting interval \([0, \pi/2]\) (use “pi” for \(\pi\)) and write down the values of \(p_1, p_5, p_{10}, p_{20}\).

9. (Math 274) Suppose we have a \(k\)-digit rounding machine. Let \(u = 5 \cdot 10^{-k}\), so the relative error of \(k\)-digit rounding of any real number is \(\leq u\). We of course take \(k \geq 1\), so note \(u < 1\).

(a) If \(x_i\) are real numbers, show \(\prod_{i=1}^{n} x_i\), performed in the machine, has relative error bounded by \((2n-1)u + O(u^2)\).

(b) If \(x_i > 0\) are positive floating point numbers, show \(((x_1 + x_2) + x_3) + x_4\), performed in the machine, has relative error bounded by \(3u + O(u^2)\). Guess an analogous result for

\[ (((((x_1 + x_2) + x_3) + x_4) + x_5) + x_6) + x_7) + x_8.\]

(c) If \(x_i > 0\) are positive floating point numbers, show \((x_1 + x_2) + (x_3 + x_4)\), performed in the machine, has relative error bounded by \(2u + O(u^2)\). Guess an analogous result for

\[ ((x_1 + x_2) + (x_3 + x_4)) + ((x_5 + x_6) + (x_7 + x_8)).\]