Homework #7

1. Consider the table of values

<table>
<thead>
<tr>
<th>x</th>
<th>x₀</th>
<th>x₁</th>
<th>x₂</th>
<th>x₃</th>
<th>x₄</th>
</tr>
</thead>
<tbody>
<tr>
<td>y₀</td>
<td>y₁</td>
<td>y₂</td>
<td>y₃</td>
<td>y₄</td>
<td></td>
</tr>
</tbody>
</table>

with \( x_i = i/4 \) and \( y_i = 4/(1 + x_i^2) \).

(a) Use composite trapezoidal rule to approximate the integral. What is the exact absolute error given that the exact value is \( \pi \)?

(b) Use instead Simpson’s rule. What is the exact absolute error?

(c) Use Simpson’s rule on the table with more values:

<table>
<thead>
<tr>
<th>x</th>
<th>x₀</th>
<th>x₁</th>
<th>x₂</th>
<th>x₃</th>
<th>x₄</th>
<th>x₅</th>
<th>x₆</th>
<th>x₇</th>
<th>x₈</th>
</tr>
</thead>
<tbody>
<tr>
<td>y₀</td>
<td>y₁</td>
<td>y₂</td>
<td>y₃</td>
<td>y₄</td>
<td>y₅</td>
<td>y₆</td>
<td>y₇</td>
<td>y₈</td>
<td></td>
</tr>
</tbody>
</table>

with \( x_i = i/8 \) and \( y_i = 4/(1 + x_i^2) \). How many times smaller is the exact absolute error compared to the results in part (b)?

2. Use Taylor series to show

\[
\int_0^h f(x) \, dx - hf(h/2) = \mathcal{O}(h^3).
\]

3. Let

\[
I = \int_a^b f(x) \, dx
\]

and let \( T(h) \) denote composite trapezoidal rule approximating \( I \) using stepsize \( h \). Assume error formula

\[
I = T(h) + C_2 h^2 + C_4 h^4 + C_6 h^6 + \ldots
\]

(a) Use Richardson extrapolation with stepsizes \( h \) and \( 2h \) to derive Simpson’s rule.

(b) (not due) Use Richardson extrapolation with stepsizes \( h, 2h, 4h \) to write out the \( \mathcal{O}(h^6) \) approximation formula.

4. Consider the ODE

\[
y' = -2ty
\]

with \( y(0) = 2 \). The exact solution is \( y(t) = 2e^{-t^2} \).

(a) Use Euler’s method with stepsize \( h = 0.5 \) to approximate \( y(1) \) and find the absolute error \( E(0.5) \) of this approximation.

(b) Use Euler’s method with stepsize \( h = 0.25 \) to approximate \( y(1) \) and find the absolute error \( E(0.25) \) of this approximation.

(c) Compute \( E(0.5)/E(0.25) \).
5. (not due) Use Trapezoid Method with stepsize $h = 0.5$ to solve the ODE

$$y' = t/y$$

for $y(2)$ given $y(1) = 2$.

6. (not due) Consider the ODE

$$y' = -2ty$$

with $y(0) = 2$. The exact solution is $y(t) = 2e^{-t^2}$.

(a) Use Midpoint method with stepsize $h = 0.5$ to approximate $y(1)$ and find the absolute error $E(0.5)$ of this approximation.

(b) Use Midpoint method with stepsize $h = 0.25$ to approximate $y(1)$ and find the absolute error $E(0.25)$ of this approximation.

(c) Compute $E(0.5)/E(0.25)$.

7. (Matlab)

(a) For a given $f(t, y)$, write a Matlab function that inputs $t, y$, and outputs the value of $f(t, y)$. Then write a Matlab function that inputs:

- $t_0$ and $w_0$;
- stepsize $h$;
- number of iterations $N$;

and uses the Midpoint method, with stepsize $h$, to solve the ODE

$$y' = f(t, y),$$

with initial value $y(t_0) = w_0$, and outputs approximation $w_N$. Write out or print out this latter function.

(b) Apply your function to the case where $f(t, y) = \sin t + y$, $t_0 = 0$, $w_0 = 1$, and $(h, N) = (0.5, 2)$ and $(h, N) = (0.05, 20)$ and $(h, N) = (0.01, 100)$. Write out or print out your results in each case.

8. (Math 274) Consider, for $\lambda > 0$,

$$y' = -\lambda y,$$

with $y(0) = w_0 \neq 0$. The exact solution is $y(t) = w_0 e^{-\lambda t}$, and $\lim_{t\to\infty} y(t) = 0$.

(a) Apply Euler’s method to the ODE and write down the approximation $w_n$ in terms of $w_0$.

(b) For what $h$ does $\lim_{n\to\infty} w_n = 0$? What happens to $\lim_{n\to\infty} w_n$ for other $h$?

(c) Similarly analyze the case $\lambda > 0$: what is $\lim_{t\to\infty} y(t)$ and when does $\lim_{n\to\infty} w_n$ satisfy similar results?