Homework #8

1. Use Euler’s method as predictor and Trapezoid Method as corrector with \( h = 0.5 \) to solve the ODE

\[ y' = \sin y \]

for \( y(1) \) given \( y(0) = 1 \).

2. Use Euler’s method with stepsize \( h = 0.25 \) to solve the system of ODE’s

\[
\begin{align*}
y' &= -z \\
z' &= y
\end{align*}
\]

for \( y(1), z(1) \) given \( y(0) = 1, z(0) = 0 \).

3. (a) Write out each step of Gaussian elimination (without pivoting) on the augmented matrix for the linear system

\[
\begin{align*}
2x - 3y + z &= 1 \\
x + y - z &= 2 \\
-4x + 4z &= -1
\end{align*}
\]

(b) Now use back substitution to solve for \( x, y, z \).

4. (a) Count the number of additions/subtractions needed to perform back substitution on \( Ux = b \), where \( U \) is upper triangular, from the formula

\[
x_i = \left[ b_i - \sum_{j=i+1}^{n} u_{ij} x_j \right] / u_{ii}.
\]

(b) Also count the number of multiplications/divisions.

5. Prove one step of Gaussian elimination (without pivoting) on an \( n \times n \) symmetric matrix \( A \) with \( a_{11} \neq 0 \) produces a matrix \( B \) such that \( B(2 : n, 2 : n) \) is symmetric.

6. (a) Give an example of a \( 2 \times 2 \) matrix that doesn’t have an \( LU \) factorization.

(b) Use Gaussian elimination (without pivoting) to find the \( LU \) factorization of the matrix

\[
A = \begin{bmatrix}
-2 & 0 & 1 & -1 \\
-1 & 2 & 0 & 1 \\
4 & -1 & -2 & -4 \\
0 & 0 & 2 & 0
\end{bmatrix}.
\]

(c) Use the \( LU \) factorization and forward and back substitution to solve the linear system \( Ax = b \), where \( b = [0, 1, 2, 3]^T \).
7. (a) Count the number of additions/subtractions and multiplications/divisions it takes to get the LU factorization (assuming it exists) of a tridiagonal \( n \times n \) matrix (taking advantage of all the zeros). Thus, how many times longer does it take to compute the LU factorization of a \( 2n \times 2n \) tridiagonal matrix compared to a \( n \times n \) tridiagonal matrix?

(b) Count the number of additions/subtractions and multiplications/divisions it takes to perform forward and back substitution to solve \( LUx = b \), using the LU factorization a tridiagonal \( n \times n \) matrix (assuming it exists and taking advantage of all the zeros).

8. (Matlab)

(a) Write a Matlab function that inputs the dimension \( n \) and a \( n \times n \) matrix \( A \), performs finds the LU factorization of \( A \) (stored in a single matrix), and outputs the number of flops used. Print out or write out your function.

(b) Apply your function to \( 10 \times 10 \), \( 20 \times 20 \), \( 100 \times 100 \), and \( 200 \times 200 \) matrices and print out your results in each case.

9. (Math 274) An \( n \times n \) matrix \( A \) is strictly diagonally dominant if

\[
|a_{ii}| > \sum_{j=1, j \neq i}^{n} |a_{ij}|
\]

for all \( i = 1, \ldots, n \). Prove Gaussian elimination (without pivoting) on a strictly diagonally dominant matrix produces a strictly diagonally dominant upper triangular matrix.