Homework #8

1. Euler’s method gives

\[ w_1^* = w_0 + h \sin w_0 = 1 + 0.5 \sin 1 = 1.42073549240395. \]

Then Trapezoid method gives

\[ w_1 = w_0 + \frac{h}{2} [\sin w_0 + \sin w_1^*] = 1 + 0.25 [\sin 1 + \sin 1.42073549240395] = 1.45755824248413. \]

Continuing, we get

\[ w_2^* = w_1 + h \sin w_1 = 1.45755824248413 + 0.5 \sin 1.45755824248413 = 1.95435595062392 \]
\[ w_2 = w_1 + \frac{h}{2} [\sin w_1 + \sin w_2^*] \]
\[ = 1.45755824248413 + 0.25 [\sin 1.45755824248413 + \sin 1.95435595062392] \]
\[ = 1.93779170093176. \]

This \( w_2 \) is the approximation for \( y(1) \).

4. (a) Note

\[ x_i = \left( b_i - \sum_{j=i+1}^{n} u_{ij} x_j \right) / u_{ii}. \]

In the numerator, \( 1 + n - i \) real numbers are being added together at one time, which requires \( n - i \) additions/subtractions. Thus, adding together all these additions/subtractions for \( i = 1, \ldots, n \), we get

\[ \sum_{i=1}^{n} n - i = \sum_{k=0}^{n-1} k, \]

where \( k = n - i \). This simplifies to \( n(n - 1)/2 \) additions/subtractions.

(b) Note

\[ x_i = \left( b_i - \sum_{j=i+1}^{n} u_{ij} x_j \right) / u_{ii}. \]

In the numerator, there are \( n - i \) multiplications. Then there is 1 division, for a total of \( n - i + 1 \) multiplications/divisions. Thus, adding together all these multiplications/divisions for \( i = 1, \ldots, n \), we get

\[ \sum_{i=1}^{n} n - i + 1 = \sum_{m=1}^{n} m, \]

where \( m = n - i + 1 \). This simplifies to \( n(n + 1)/2 \) multiplications/divisions.
5. A symmetric means $a_{ij} = a_{ji}$ for all $i, j = 1, \ldots, n$. We will prove $b_{ij} = b_{ji}$ for all $i, j = 2, \ldots, n$. We get

\[
b_{ij} = a_{ij} - \frac{a_{1j}}{a_{11}}a_{i1} = a_{ji} - \frac{a_{j1}}{a_{11}}a_{1i} = a_{ji} - \frac{a_{j1}}{a_{11}}a_{1i} = b_{ji}.
\]

8. (Matlab)

(a) See “hw8afn.m”.

(b) For $n = 10$, we get 615 flops; for $n = 20$, we get 5130 flops; for $n = 100$, we get 661650 flops; and for $n = 200$, we get 5313300 flops.

9. (Math 274) We first prove the result for the first step of Gaussian elimination. Let $B$ be the matrix after the first step of Gaussian elimination. Note $b_{1j} = a_{1j}$, for all $j$, so

\[
\sum_{j=2}^{n} |b_{1j}| = \sum_{j=2}^{n} |a_{1j}| < |a_{11}| = |b_{11}|
\]

For $i = 2, \ldots, n$, note $b_{1i} = 0$. So

\[
\sum_{j=2, j \neq i}^{n} |b_{ij}| = \sum_{j=2, j \neq i}^{n} |b_{ij}|
\]

\[
= \sum_{j=2, j \neq i}^{n} \left| a_{ij} - \frac{a_{i1}}{a_{11}}a_{1j} \right|
\]

\[
\leq \sum_{j=2, j \neq i}^{n} \left| a_{ij} \right| + \left| \frac{a_{i1}}{a_{11}} \right| \sum_{j=2, j \neq i}^{n} |a_{1j}|
\]

\[
= \sum_{j=2, j \neq i}^{n} |a_{ij}| + \left| \frac{a_{i1}}{a_{11}} \right| \sum_{j=2, j \neq i}^{n} |a_{1j}|
\]

\[
= \sum_{j=1, j \neq i}^{n} |a_{ij}| - |a_{i1}| + \left| \frac{a_{i1}}{a_{11}} \right| \left( \sum_{j=2}^{n} |a_{1j}| - |a_{1i}| \right)
\]

\[
< |a_{ii}| - |a_{i1}| + \left| \frac{a_{i1}}{a_{11}} \right| (|a_{11}| - |a_{1i}|)
\]

\[
= |a_{ii}| - \left| \frac{a_{i1}}{a_{11}} \right| |a_{1i}|
\]

\[
\leq |a_{ii} - \frac{a_{i1}}{a_{11}} a_{1i}|
\]

\[
= |b_{ii}|
\]
This proves given a matrix that is strictly diagonally dominant, after one step of Gaussian elimination, the resulting matrix is still strictly diagonally dominant. So, by induction, note, for the base case, $A$ is strictly diagonally dominant. Now suppose $A^{(k)}$, the result of $k$ steps of Gaussian elimination on $A$, is strictly diagonally dominant. Then $A^{(k+1)}$ is also strictly diagonally dominant since it is one step of Gaussian elimination on $A^{(k)}$. Therefore, by induction, $A^{(k)}$ is strictly diagonally dominant for all $k = 0, \ldots, n - 1$. Thus, the upper triangular matrix we are interested in, $A^{(n-1)}$, is strictly diagonally dominant.