3. Proof by induction.

Base case: $A^{(1)} = A$, which is strictly diagonally dominant.

Suppose $A^{(k)}$ is strictly diagonally dominant, then we have for $i \geq k$, $a_{ij}^k = 0$ for $j \leq k - 1$, and

$$a_{ii}^k > \sum_{j=k, j \neq i}^n |a_{ij}^k|$$

We need to show for $i \geq k + 1$

$$a_{ii}^{k+1} > \sum_{j=k+1, j \neq i}^n |a_{ij}^{k+1}|$$

$$\sum_{j=1, j \neq i}^n |a_{ij}^{k+1}| = \sum_{j=k+1, j \neq i}^n |a_{ij}^k - a_{ik}^k a_{kj}^k/a_{kk}^k|$$

by GE

$$\leq \sum_{j=k+1, j \neq i}^n |a_{ij}^k| + \frac{|a_{ik}^k|}{|a_{kk}^k|} |a_{kj}^k|$$

triangle inequality

$$\leq |a_{ii}^k| - |a_{ik}^k| + \frac{|a_{ik}^k|}{|a_{kk}^k|} (|a_{kj}^k| - |a_{ki}^k|)$$

induction hypothesis

$$= |a_{ii}^k| - |a_{ik}^k|/|a_{kk}^k| |a_{ki}^k|$$

this is positive since $|a_{ii}^k| > |a_{ik}^k|, |a_{kk}^k| > |a_{ki}^k|$ reverse triangle inequality

$$\leq |a_{ii}^k| - \frac{a_{ik}^k}{a_{kk}^k} |a_{ki}^k|$$

Note:

- 1 improper induction (e.g., show $k=2$, and “similarly, by induction ...”)
- 1 missing base case
- 1 minor mistake in inequalities
- 2 major mistake in inequalities (e.g., ignoring absolute values)
5(a) Since $L$ and $M$ are lower triangular, for $j > i$, $l_{ij} = m_{ij} = 0$, and therefore

$$[LM]_{ij} = \sum_{k=1}^{n} l_{ik}m_{kj} = \sum_{k=1}^{j-1} l_{ik}m_{kj} + \sum_{k=j}^{n} l_{ik}m_{kj} = 0.$$ 

On the diagonal, we have $[LM]_{ii} = l_{ii}m_{ii}$. Therefore, $LM$ is also lower triangular.

(b) Proof by induction. Base case: for $n = 1$, $L = l_{11} \in \mathbb{R}$, $L^{-1} = l_{11}^{-1}$, which is lower triangular. The diagonal of $L^{-1}$ is the reciprocal of $L$. Suppose the inverse of nonsingular lower triangular $L \in \mathbb{R}^{n \times n}$ is lower triangular, and the diagonals are the corresponding reciprocal of the diagonals of $L$. For $L \in \mathbb{R}^{(n+1) \times (n+1)}$, partition $L$ as

$$\begin{bmatrix}
\hat{L} & 0 \\
\hat{r}^T & l_{nn}
\end{bmatrix},$$

where $\hat{L} \in \mathbb{R}^{n \times n}$. Partition conformally $L^{-1} = \begin{bmatrix} A & b \\ c^T & d \end{bmatrix}$. Since $LL^{-1} = I$, we have

$$LL^{-1} = \begin{bmatrix}
\hat{L}A & \hat{L}b \\
\hat{r}^TA + l_{nn}c^T & l_{nn}d
\end{bmatrix} = \begin{bmatrix} I_n & 0 \\ 0 & 1 \end{bmatrix}.$$ 

By induction hypothesis, $A = \hat{L}^{-1}, b = 0$, and $d = l_{nn}^{-1}$. Therefore $L^{-1}$ is also lower triangular, and its diagonal elements are the reciprocal of those of $L$.

Another approach is to partition $L^{-1} = [y_1, y_2, \ldots, y_n]$, and perform forward substitution on $Ly_i = e_i$.

(c) In LU factorization $A = LU$, where $L$ is unit triangular. Suppose there are nonunique LU factorization of $A = L_1U_1 = L_2U_2$. Then $L_2^{-1}L_1 = U_2U_1^{-1}$. From (b), the left hand side is a unit lower triangular matrix, the right hand side is an upper triangular matrix. Therefore, it must be the case that $L_2^{-1}L_1 = U_2U_1^{-1} = I$. Thus $L_2 = L_1, U_2 = U_1$. The factorization is unique.

(d) Consider the LU factorization $A = LU$, where $L$ is unit lower triangular and diagonals of $U$ is nonzero. We can factor $U = DU'$, where $D$ is the diagonal of $U$, a diagonal matrix, and $U'$ is given by dividing the i-th row of $U$ by $u_{ii}$, hence a unit upper triangular matrix. (Another way is to consider LU factorization of $U^T$.) Then $A = LDL'$. Since $A$ is symmetric, $A = A^T = U^TDL^T$. By uniqueness of LU decomposition, we have $U' = L$.

Note:

-1 improper induction (e.g., show $k=2$, and “similarly, by induction ...”)

Avoid Matlab notation such as $A(1:i, 1:j)$ except for matrix partition.