Homework #3

For the following problems, we ignore overflow and underflow in machine numbers.

1. Let $x, y \in \mathbb{R}$.

   (a) Show, when $x \neq 0$, the relative error of transforming $x$ into its closest machine number, in a base $10$, $k$-digit rounding machine,
   $$\frac{|x - fl(x)|}{|x|},$$
   is $\leq 0.5 \cdot 10^{1-k}$, for any $k \geq 1$.

   (b) Given $k \geq 1$, find an example of $x, y \in \mathbb{R}$ such that $x + y$, performed in a base $10$, $k$-digit rounding machine, has relative error $= 1$.

   (c) Show, when $x, y \neq 0$, the relative error of $x \cdot y$ performed in a machine with unit roundoff error $u$,
   $$\frac{|x \cdot y - fl(fl(x) \cdot fl(y))|}{|x \cdot y|},$$
   is $\leq 3u + \mathcal{O}(u^2)$.

2. Let $u$ be the unit roundoff error in a machine.

   (a) Show, for $x_1, x_2, x_3, x_4 \in \mathbb{R}$, the machine version of $((x_1 + x_2) + x_3) + x_4$ can be written as $\hat{x}_1 + \hat{x}_2 + \hat{x}_3 + \hat{x}_4$, where $\hat{x}_i = x_i(1 + \epsilon_i)$ with $|\epsilon_i| \leq 4u + \mathcal{O}(u^2)$.

   (b) Show, for $x_1, x_2, x_3, x_4 \in \mathbb{R}$, the machine version of $(x_1 + x_2) + (x_3 + x_4)$ can be written as $\hat{y}_1 + \hat{y}_2 + \hat{y}_3 + \hat{y}_4$, where $\hat{y}_i = x_i(1 + \delta_i)$ with $|\delta_i| \leq 3u + \mathcal{O}(u^2)$.

   (c) Guess similar bounds for $\epsilon_i$ and $\delta_i$, when computing $\sum_{i=1}^{2^n} x_i$, for $x_i \in \mathbb{R}, 1 \leq i \leq 2^n$, using the two strategies above. Which is better?

   (To clarify, the second strategy for 8 numbers performs $((x_1 + x_2) + (x_3 + x_4)) + ((x_5 + x_6) + (x_7 + x_8))$)

3. The LU factorization of an $n \times n$ matrix $A$ of machine numbers is computed, in the Doolittle algorithm, using the formulas

   $$l_{ij} = \frac{a_{ij} - \sum_{k=1}^{j-1} l_{ik} u_{kj}}{u_{jj}},$$
   for $i > j$, and

   $$u_{ij} = a_{ij} - \sum_{k=1}^{i-1} l_{ik} u_{kj},$$
   for $i \leq j$. In a machine with unit roundoff error $u$, $\hat{L} = (\hat{l}_{ij})$ and $\hat{U} = (\hat{u}_{ij})$ are computed instead. Let $A + E = \hat{L}\hat{U}$, where $E = (e_{ij})$. 

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(a) For $i > j$, find $C_1(n)$ such that

$$|e_{ij}| \leq C_1(n)u \sum_{k=1}^{j} |\hat{L}_{ik}| |\hat{U}_{kj}| + O(u^2).$$

(Note, your answers will depend on how you order your summations)

(b) For $i \leq j$, find $C_2(n)$ such that

$$|e_{ij}| \leq C_2(n)u \sum_{k=1}^{i} |\hat{L}_{ik}| |\hat{U}_{kj}| + O(u^2).$$

(c) What does this mean for $|E|$ and $||E||_\infty$?

4. The approximate solution $x + \delta x$ obtained when solving, in a machine with unit roundoff error $u$, $Ax = b$, for machine-valued $n \times n$ matrix $A$ and $n \times 1$ vector $b$, by LU factorization followed by forward and back substitution, satisfies $(A + \delta A)(x + \delta x) = b$. Suppose we know we can get that forward or back substitution on $Bz = c$, for machine-valued $n \times n$ matrix $B$ and $n \times 1$ vector $c$, in the machine produces approximation $z + \delta z$ satisfying $(B + \delta B)(z + \delta z) = c$, where

$$|\delta B| \leq \max \{ (n - 1), 2 \} u |B| + O(u^2).$$

Use this result, along with the results of your previous problem, to find $C(n)$ satisfying

$$|\delta A| \leq C(n)u|\hat{L}| |\hat{U}| + O(u^2).$$

What does this mean for $||\delta A||_\infty/||A||_\infty$?

5. Let $A \in \mathbb{R}^{n \times n}$ be symmetric, positive definite.

(a) Prove $A(1 : k, 1 : k)$ and $A(k + 1 : n, k + 1 : n)$ are symmetric, positive definite.

(b) Let $A^{(k)}$ denote the matrix before the $k$th step of Gaussian elimination without pivoting. Explain why $A^{(2)}$ exists. Then prove $A^{(2)}(2 : n, 2 : n)$ is symmetric positive definite. What does this mean for $A^{(k)}(k : n, k : n)$, for $2 \leq k \leq n$?

(c) Conclude $\det A(1 : k, 1 : k) > 0$, for all $1 \leq k \leq n$.

6. Let $A \in \mathbb{R}^{n \times n}$ be a symmetric matrix.

(a) Prove if $\det A(1 : k, 1 : k) > 0$, for all $1 \leq k \leq n$, then $A = LDL^T$, for some unit triangular $L$, and diagonal matrix $D$ with positive diagonal elements.

(b) Conclude that $A$ is symmetric, positive definite.