Homework #4

1. Let $A \in \mathbb{R}^{n \times n}$ be symmetric, positive definite.
   (a) Given $X$ nonsingular, prove $X^TAX$ is symmetric positive definite.
   (b) Given $1 \leq i,j \leq n$, $i \neq j$, find a permutation matrix $X$ such that $B = X^TAX$ satisfies
   $$B(1:2, 1:2) = \begin{bmatrix} a_{ii} & a_{ij} \\ a_{ji} & a_{jj} \end{bmatrix}.$$ 
   Conclude that $|a_{ij}| < \max\{a_{ii}, a_{jj}\}$, for all $1 \leq i,j \leq n$.
   (c) Prove the $LU$ factorization of $A$, $A = LU$, satisfies
   $$|u_{ij}| \leq \max_{1 \leq r \leq n} \{a_{rr}\},$$
   for all $1 \leq i,j \leq n$. Conclude that $||U||_\infty / ||A||_\infty \leq n$.

2. Let $A \in \mathbb{R}^{n \times n}$ be a nonsingular matrix with an $LU$ factorization. Suppose $A$ has lower bandwidth $r$ and upper bandwidth $s$. Show the number of flops needed to compute the $LU$ factorization can be $O(nrs)$.

3. Write a Matlab program that computes the $LU$ factorization, in the form of stored diagonals, given a tridiagonal matrix in the form of three vectors, one representing the main diagonal, one representing the upper diagonal, and one representing the lower diagonal.

4. (a) Let $L \in \mathbb{R}^{n \times n}$ be a lower triangular, nonsingular matrix, and let $x,b \in \mathbb{R}$ such that $Lx = b$. Suppose we have the partitions
   $$L = \begin{bmatrix} L_{11} & L_{12} \\ L_{21} & L_{22} \end{bmatrix}, b = \begin{bmatrix} v \\ w \end{bmatrix}, x = \begin{bmatrix} y \\ z \end{bmatrix},$$
   where $L_{11} \in \mathbb{R}^{k \times k}$, $v,y \in \mathbb{R}^k$, for some $1 \leq k \leq n$, and suppose $v = 0$. Conclude $y = 0$. Furthermore, if the first component of $w$ is nonzero, conclude the first component of $z$ is also nonzero.
   (b) Let $A \in \mathbb{R}^{n \times n}$ be a symmetric, positive definite matrix with partition,
   $$A = \begin{bmatrix} A_{11} & v & A_{13} \\ c^T & a & d^T \\ A_{31} & w & A_{33} \end{bmatrix},$$
   where $A_{11} \in \mathbb{R}^{k \times k}$, $v \in \mathbb{R}^k$, $a \in \mathbb{R}$, for some $1 \leq k \leq n - 1$. Let $R = L^T$, where $L$ is the Cholesky factor of $A$, and define $l_j(B) = \min \{i | b_{ij} \neq 0\}$, for $1 \leq j \leq n$, for any matrix $B \in \mathbb{R}^{n \times n}$ with nonzero diagonal. Use the same matrix partition on $R$ to show $l_{k+1}(R) = l_{k+1}(A)$.

5. Let $A \in \mathbb{R}^{n \times n}$ be nonsingular, with nonzero diagonal elements, and suppose $A$ has an $LU$ factorization, $A = LU$. Define
   $$f_i(B) = \min \{j | b_{ij} \neq 0\},$$
   $$l_j(B) = \min \{i | b_{ij} \neq 0\},$$
for $1 \leq i, j \leq n$, for any matrix $B \in \mathbb{R}^{n \times n}$ with nonzero diagonal elements. Prove $f_i(A) = f_i(L)$ and $l_j(A) = l_j(U)$, for $1 \leq i, j \leq n$.

6. Let $A \in \mathbb{R}^{n \times n}$ be nonsingular. Suppose we know, for any $B \in \mathbb{R}^{n \times n}$, $\|B\|_2 = \sigma_1(B)$, where $\sigma_1(B)$ denotes the largest singular value of $B$. Prove $\kappa_2(A^TA) = \kappa_2(A^2)$. What does this mean for solutions to perturbations of the linear system $Ax = b$ versus those of $A^T Ax = c$?

7. Let $A \in \mathbb{R}^{m \times n}$ be a matrix of rank $n$. Find the form the matrix, in $\mathbb{R}^{m \times n}$, of rank $k < n$, in terms of the SVD of $A$, that best approximates $A$ under the 2-norm.

You may use the fact that if $B$ is rank $k$, then $\sigma_1(A - B) \geq \sigma_{k+1}(A)$, where $\sigma_1(C) \geq \ldots \geq \sigma_n(C)$ denote the singular values of $C$. 

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