270A HW5 graded questions

4. (credit to Yanyi Wang)
   (a) Let $A = QR$. Let $A = [a_1 ... a_n]$ and $R = [r_1 ... r_n]$. 

   \[ |\det(A)| = |\det(Q)||\det(R)| = |\det(R)| = \prod_{i=1}^{n} |r_{ii}|, \]

   as $Q$ is orthogonal, $|\det(Q)| = 1$ and $R$ is upper triangular. Since $A^T A = R^T R$, by considering the diagonal entries, we have $\|a_i\|_2^2 = \|r_{ii}\|_2^2$ for $i = 1, ..., n$. Notice that $\|r_{ii}\|_2 \geq |r_{ii}|$. Therefore \[ |\det(A)| \leq \prod_{i=1}^{n} \|a_i\|_2. \]

   (b) $|\det(A)| = \prod_{i=1}^{n} \|a_i\|_2$ implies that \[ \prod_{i=1}^{n} \|r_{ii}\|_2 = \prod_{i=1}^{n} r_{ii}^2. \]

   Since $\|r_{ii}\|_2^2 \geq r_{ii}^2$, the equation hold if and only if $r_{ii} = |0, ..., 0, r_{ii}, 0, ..., 0|^T$, i.e. the i-th entry of $r_{ii}$ is $r_{ii}$ and the rest are 0. Therefore, if there exist i such that $r_{ii} = 0$, then $\|a_i\|_2 = 0$ and $A$ has a zero column $a_i$. If $r_{ii} \neq 0$ for all i, then $A^T A = \text{diag}(|a_1|_2^2, ..., |a_n|_2^2)$, a diagonal matrix.

5. (a) Let $Q = [q_1 ... q_n]$, $B = PQ = [Pq_1 ... Pq_n]$. The first column of $B = PQ_1 = e_1$. For $2 \leq i \leq n$, 

   \[ B_{ii} = e_1^T (Pq_i) = (Pq_1)^T (Pq_i) = q_i^T q_1 = 0 \]

   we have the transpose of the first row of $B = e_1$. Partition $B$ as \[ \begin{bmatrix} 1 & 0 \\ 0 & B \end{bmatrix}. \] Since $B^T B = Q^T Q = I$, we have $B^T \hat{B} = I$. Thus $B(2 : n, 2 : n)$ is orthogonal.

   (b) We prove the statement by induction. For $n=1$, $Q = \pm 1$. Suppose for $Q \in \mathbb{R}^{(n-1) \times (n-1)}$, there exists orthogonal $P \in \mathbb{R}^{(n-1) \times (n-1)}$ such that $PQ = D$, where $D$ is a diagonal matrix with $d_{ii} = 1$, for all $1 \leq i \leq n-1$, and $d_{nn} = \pm 1$. For $Q \in \mathbb{R}^{n \times n}$, from (a), we know that there exist $P' \in \mathbb{R}^{n \times n}$ such that $P'Q = \begin{bmatrix} 1 & 0 \\ 0 & Q' \end{bmatrix}$. By induction hypothesis, there exist $P \in \mathbb{R}^{(n-1) \times (n-1)}$ such that $PQ = \text{diag}(1, 1, ..., \pm 1) \in \mathbb{R}^{(n-1) \times (n-1)}$. Let $P = \begin{bmatrix} 1 & 0 \\ 0 & P' \end{bmatrix}$. Then $P$ is orthogonal, and $PQ = \text{diag}(1, 1, ..., \pm 1) \in \mathbb{R}^{n \times n}$. 
