Homework #6

1. Let $A \in \mathbb{R}^{m \times n}$ with $\text{rank}(A) = n$. Given the thin QR factorization, $A = Q_1 R_1$, where $Q_1 \in \mathbb{R}^{m \times n}$ has orthonormal columns and $R_1 \in \mathbb{R}^{n \times n}$ is upper triangular, explain how to use $Q_1, R_1$ to solve the least squares problem $\min ||b - Ax||_2$.

2. (a) Let $A \in \mathbb{R}^{m \times n}$ and let $H$ be a plane reflector such that $B = HA$ satisfies $b_{1j} = 0$, for $2 \leq i \leq m$. Suppose $a_{k1} = 0$, for some $2 \leq k \leq m$. Prove $b_{kj} = a_{kj}$ for all $1 \leq j \leq n$.

   (b) Let $p$ = the number of nonzero elements in the first column of $A$. Suppose we have $v \in \mathbb{R}^m$ such that $H = I - 2vv^T$. Show that we can calculate $B(1 : m, 2 : n)$ in $4p(n-1)$ flops.

   (c) Give an example of $A \in \mathbb{R}^{2 \times 2}$ such that in the Householder QR factorization of $A$, $A = QR$, the upper bandwidth of $R$ is greater than that of $A$.

3. Let $x = [r \cos \theta, r \sin \theta]^T$, for some $r > 0$ and $0 \leq \theta < 2\pi$. Also, let $H$ be the plane reflection such that $Hx = [r, 0]^T$ and let $G$ be the plane rotation such that $Gx = [r, 0]^T$.

   (a) Prove, for any $y \times \mathbb{R}^2$, $(Gy)_2 = -(Hy)_2$. If the first column of nonsingular $A \in \mathbb{R}^{2 \times 2}$ is $x$, what does this mean for $R$ and $U$, where $HA = R$ and $GA = U$?

   (b) Prove, in fact, $(Gy)_1 = (Hy)_1$. Thus, find a nonsingular matrix $B \in \mathbb{R}^{2 \times 2}$ such that $H = BG$.

4. Consider the plane rotation matrix,

$$G = \begin{bmatrix} c & s \\ -s & c \end{bmatrix} \in \mathbb{R}^{2 \times 2}$$

and let $v \in \mathbb{R}^2$.

   (a) Count exactly the number of additions/subtractions and multiplications/divisions needed to calculate $Gv$.

   (b) Show, for $p = \frac{s}{1 + c}$, if

$$w = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} cv_1 + sv_2 \\ -p(v_1 + w_1) + v_2 \end{bmatrix},$$

then $w = Gv$.

   (c) Count exactly the number of additions/subtractions and multiplications/divisions needed to calculate $w$.

5. (a) Let $A \in \mathbb{R}^{m \times n}$ and let $G_{ij} \in \mathbb{R}^{n \times n}$, for $i < j$, denote a plane rotation in the $(i,j)$-plane. Explain what happens, in terms of rotations, when you multiply $A$ on the right by $G_{ij}$.
(b) For $A \in \mathbb{R}^{3 \times 3}$ upper triangular, explain how to choose plane rotations $G_{12}$ and $\hat{G}_{12}$, both in the $(1,2)$-plane, such that $(G_{12}A)\hat{G}_{12}$ has lower bandwidth 0 and upper bandwidth 1.

(c) How would you modify this approach to transform upper triangular $A \in \mathbb{R}^{n \times n}$ to have lower bandwidth 0 and upper bandwidth 1?

6. Let $A \in \mathbb{R}^{m \times n}$, where $A$ has linearly independent rows. Explain how you can get the factorization $A = R_1Q_1$, where $R_1 \in \mathbb{R}^{m \times m}$ is upper triangular and $Q_1 \in \mathbb{R}^{m \times n}$ has orthonormal rows.

7. Consider the square root-free modified Gram-Schmidt algorithm that defines, for given linearly independent $a_1, \ldots, a_n \in \mathbb{R}^m$, with $m \geq n$, orthogonal $q_1, \ldots, q_n$ by: $q_1 = a_1$ and

$$q_k = (I - q_{k-1}q_{k-1}^T) \cdots (I - q_1q_1^T)a_k,$$

for $2 \leq k \leq n$. Let $A = [a_1 \ldots a_n]$ and $Q_1 = [q_1 \ldots q_n]$. Prove $A = Q_1R_1$, for some unit upper triangular $R_1 \in \mathbb{R}^{n \times n}$. 

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