Homework #8

1. Let \( A \in \mathbb{R}^{n \times n} \) satisfy
   \begin{itemize}
   \item \( a_{ii} = \beta \in \mathbb{R} \), for \( 1 \leq i \leq n \);
   \item \( a_{i,i+1} = 1 \), for \( 1 \leq i \leq n-1 \);
   \item \( a_{ij} = 0 \), otherwise.
   \end{itemize}

   Let \( E = \tau e_n e_1^T \), for \( \tau \in \mathbb{R} \).

   (a) Find the eigenvalues of \( A \) and \( A + E \).

   (b) Let \( \lambda \) be an eigenvalue of \( A + E \). Find an example of \( n \) such that even with \( \tau = 10^{-14} \), \( |\lambda - \beta| = 10^{-1} \). What is \( |\lambda - \beta| \) when \( n = 100 \)?

2. Let \( A, A + E \in \mathbb{R}^{n \times n} \) be symmetric matrices. Let
   \[ \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_n \]
denote the eigenvalues of \( A \),
   \[ \beta_1 \geq \beta_2 \geq \ldots \geq \beta_n \]
denote the eigenvalues of \( E \), and
   \[ \gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_n \]
denote the eigenvalues of \( A + E \).

   (a) Use the fact (from eigenvalue interlacing theorems for symmetric matrices) that
   \[ \alpha_j + \beta_1 \geq \gamma_j \geq \alpha_j + \beta_n, \]
   for all \( 1 \leq j \leq n \), to show \( |\alpha_j - \gamma_j| \leq \|E\|_2 \), for all \( 1 \leq j \leq n \).

   (b) In addition, use Geršgorin’s Theorem to show \( |\alpha_j - \gamma_j| \leq \|E\|_\infty \), for all \( 1 \leq j \leq n \).

3. Given \( A \in \mathbb{R}^{n \times n} \), consider the QR algorithm: \( A_1 = A \) and \( A_{k+1} = R_k Q_k \), for \( k \geq 0 \), where \( Q_k R_k \) is a QR factorization of \( A_k \). Prove
   \[ A^k = Q_1 \cdots Q_k R_k \cdots R_1, \]
   for all \( k \geq 0 \).

4. Given \( R \in \mathbb{R}^{n \times n} \) upper triangular, and \( G_{i,i+1} \) plane rotators in the \((i, i+1)\)-plane, prove \( RG_{1,2}^T \cdots G_{n-1,n}^T \) is upper Hessenberg. What does this mean for the matrix at each step of the QR algorithm on an upper Hessenberg matrix, where the QR factorization is computed with plane rotators?

5. Let \( A \in \mathbb{R}^{n \times n} \) be symmetric and suppose there exists \( Q \in \mathbb{R}^{n \times n} \) orthogonal such that \( Q^T AQ \) is upper Hessenberg. Prove \( Q^T AQ \) is tridiagonal.
6. Let $A \in \mathbb{R}^{n \times n}$ is an irreducible ($a_{i+1,i} \neq 0$ for all $1 \leq i \leq n - 1$) upper Hessenberg matrix.

(a) If $A$ is singular, prove the existence of a QR factorization satisfying $r_{nn} = 0$.

(b) Consider one step of a QR algorithm with a shift $\mu$ on $A$: $A_1 = A - \mu I$ and $A_2 = R_1Q_1 + \mu I$, where $Q_1R_1$ is the QR factorization (as above) of $A_1$. Prove if $\mu$ is an eigenvalue of $A$, then $(A_2)_{nn} = \mu$. 