Homework #1

Turn in the problems marked with a (*)

1. Suppose \( q(x) \) is the interpolation polynomial for \((x_0, y_0), \ldots, (x_{n-1}, y_{n-1})\). Then the interpolation polynomial for \((x_0, y_0), \ldots, (x_n, y_n)\) is, under Newton’s form, 

\[
p(x) = q(x) + C(x - x_0) \cdot \ldots \cdot (x - x_{n-1})
\]

a) Identify the coefficient of the \(x^n\) term in the polynomial \(p(x)\).

The usual notation for \(C\) is \(f[x_0, \ldots, x_n]\)

b) The Lagrange form for the interpolation polynomial writes \(p(x)\) as

\[
p(x) = \sum_{i=0}^{n} y_i \prod_{j=0, j \neq i}^{n} \frac{x - x_j}{x_i - x_j}.
\]

Use this to write down a formula for \(C\).

c) Let \(r(x)\) be the interpolation polynomial for the data points \((x_1, y_1), \ldots, (x_n, y_n)\).

Prove

\[
p(x) = \frac{(x - x_0)r(x) - (x - x_n)q(x)}{x_n - x_0}
\]

then use this to write down a formula for \(C\) in terms of the coefficients of the \(x^{n-1}\) terms of the polynomials \(q(x)\) and \(r(x)\) (with usual notation \(f[x_0, \ldots, x_{n-1}]\) and \(f[x_1, \ldots, x_n]\), respectively).

2. (*) Write down the Newton form for the interpolation polynomial for the data points

\((0, 1), (2, 2), (3, 1), (5, 0)\)

by forming the following divided difference table and using its values (using the results of 1(c)).

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>2</th>
<th>3</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>(f[0])</td>
<td>(f[0, 2])</td>
<td>(f[0, 2, 3])</td>
<td>(f[0, 2, 3, 5])</td>
</tr>
<tr>
<td>2</td>
<td>(f[2])</td>
<td>(f[2, 3])</td>
<td>(f[2, 3, 5])</td>
<td>(f[3, 5])</td>
</tr>
<tr>
<td>3</td>
<td>(f[3])</td>
<td>(f[3, 5])</td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>(f[5])</td>
<td></td>
<td></td>
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</tr>
</tbody>
</table>

3. Let

\(f \in C^4[a, b] = \{\text{set of functions where } f, f', f'', f''', f'''' \text{ exist and are continuous in } [a, b]\}\).

Let \((x_0, f(x_0)), (x_1, f(x_1)), \ldots, (x_n, f(x_n))\) be \(n+1\) data points, sampled from the graph of \(f\), with

\[a = x_0 < x_1 < \ldots < x_n = b,\]
and $x_{j+1} - x_j = h$, for all $j = 0, \ldots, n-1$. Denote by $p_i(x)$ the interpolation polynomial using data points

$$(x_{i-1}, f(x_{i-1})), (x_i, f(x_i)), (x_{i+1}, f(x_{i+1})), (x_{i+2}, f(x_{i+2})),$$

for $i = 1, \ldots, n-2$. Prove there exists a constant $C_n$ such that $|f(x) - p_i(x)| \leq C_n h^4$ for $x \in [x_i, x_{i+1}]$, for all $i = 1, \ldots, n-2$.

4. Consider the data points $(-1, 0), (0, 0), (1, 0), (2, 1), (3, 1), (4, 1)$.

(a) Sketch the two choices of polynomials involved in ENO interpolating polynomial using $(1, 0), (2, 2)$ and one more node.

(b) Continuing, sketch the two choices of polynomials involved in ENO interpolating polynomial adding yet another node.

5. (*) Consider the data points $(-1, 1), (0, 1), (1, 0), (2, 2), (3, 3), (4, 2)$.

(a) Find the ENO interpolating polynomial using $(1, 0), (2, 2)$ and one more node.

(b) Find the ENO interpolating polynomial using $(1, 0), (2, 2)$ and two more nodes.

6. Program a function for ENO interpolation that inputs

- $n$;
- data points $(x_0, y_0), \ldots, (x_n, y_n)$, with $x_0 < \ldots < x_n$;
- $x$, a location satisfying $x_0 \leq x \leq x_n$;
- $i$, the index satisfying $x \in [x_i, x_{i+1}]$;
- $p \geq 2$, the number of nodes to use;

and outputs the ENO interpolation polynomial evaluated at $x$. 