Homework #1

Turn in the problems marked with a (*).

1. Suppose $q(x)$ is the interpolation polynomial for $(x_0, y_0), \ldots, (x_{n-1}, y_{n-1})$. Then the interpolation polynomial for $(x_0, y_0), \ldots, (x_n, y_n)$ is, under Newton's form,

$$p(x) = q(x) + C(x - x_0) \cdot \ldots \cdot (x - x_{n-1}).$$

   (a) Identify the coefficient of the x^n term in the polynomial $p(x)$.
   (The usual notation for C is $f[x_0, \ldots, x_n]$)

   (b) The Lagrange form for the interpolation polynomial writes $p(x)$ as

$$p(x) = \sum_{i=0}^{n} y_i \prod_{j=0, j\neq i}^{n} \frac{x - x_j}{x_i - x_j}.$$

   Use this to write down a formula for C.

   (c) Let $r(x)$ be the interpolation polynomial for the data points $(x_1, y_1), \ldots, (x_n, y_n)$. Prove

$$p(x) = \frac{(x - x_0)r(x) - (x - x_n)q(x)}{x_n - x_0},$$

   then use this to write down a formula for C in terms of the coefficients of the x^{n-1} terms of the polynomials $q(x)$ and $r(x)$ (with usual notation $f[x_0, \ldots, x_{n-1}]$ and $f[x_1, \ldots, x_n]$, respectively).

2. (*) Write down the Newton form for the interpolation polynomial for the data points

$$(0, 1), (2, 2), (3, 1), (5, 0)$$

   by forming the following divided difference table and using its values (using the results of 1(c)).

$$\begin{array}{llll}
0 & f[0] & & \\
 & & f[0,2] & \\
2 & f[2] & & f[0,2,3] \\
 & & f[2,3] & \\
3 & f[3] & & f[2,3,5] \\
 & & f[3,5] & \\
5 & f[5] & &
\end{array}$$

3. Let

$$f \in C^4[a, b] = \{\text{set of functions where } f, f', f'', f''', f'''' \text{ exist and are continuous in } [a, b]\}.$$

   Let $(x_0, f(x_0), (x_1, f(x_1)), \ldots, (x_n, f(x_n))$ be $n+1$ data points, sampled from the graph of f, with

$$a = x_0 < x_1 < \ldots < x_n = b,$$

and $x_{j+1} - x_j = h$, for all $j = 0, \ldots, n-1$. Denote by $p_i(x)$ the interpolation polynomial using data points

$$(x_{i-1}, f(x_{i-1})), (x_i, f(x_i)), (x_{i+1}, f(x_{i+1})), (x_{i+2}, f(x_{i+2})),$$

for $i = 1, \ldots, n-2$. Prove there exists a constant C_n such that $|f(x) - p_i(x)| \leq C_n h^4$ for $x \in [x_i, x_{i+1}]$, for all $i = 1, \ldots, n-2$.

4. Consider the data points $(-1, 0), (0, 0), (1, 0), (2, 1), (3, 1), (4, 1)$.

   (a) Sketch the two choices of polynomials involved in ENO interpolating polynomial using $(1, 0), (2, 2)$ and one more node.

   (b) Continuing, sketch the two choices of polynomials involved in ENO interpolating polynomial adding yet another node.

5. (*) Consider the data points $(-1, 1), (0, 1), (1, 0), (2, 2), (3, 3), (4, 2)$.

   (a) Find the ENO interpolating polynomial using $(1, 0), (2, 2)$ and one more node.

   (b) Find the ENO interpolating polynomial using $(1, 0), (2, 2)$ and two more nodes.

6. Program a function for ENO interpolation that inputs

   - n;
   - data points $(x_0, y_0), \ldots, (x_n, y_n)$, with $x_0 < \ldots < x_n$;
   - x, a location satisfying $x_0 \leq x \leq x_n$;
   - i, the index satisfying $x \in [x_i, x_{i+1}]$;
   - $p \geq 2$, the number of nodes to use;

   and outputs the ENO interpolation polynomial evaluated at x.