Homework #4

Turn in the problems marked with a (*).

1. (*) Show, in the space of complex numbers,

\[ \sum_{k=0}^{n} a_k (z - z_0)^k = \sum_{m=0}^{n} \left( \sum_{k=m}^{n} a_k \binom{k}{m} (-z_0)^{k-m} \right) z^m, \]

where \( \binom{k}{m} \) are the binomial coefficients, satisfying

\[ \binom{k}{m} = \frac{k!}{m!(k-m)!}, \]

or

\[ (1 + x)^k = \sum_{m=0}^{k} \binom{k}{m} x^m. \]

2. (*) Show, for \(|z_0| < R\) and \(|z| > R\), for some \(R > 0\):

(a) \[ \log(z - z_0) = \log(z) - \sum_{k=1}^{\infty} \frac{1}{k} \left( \frac{z_0}{z} \right)^k. \]

(b) \[ \left| \sum_{k=p+1}^{\infty} \frac{1}{k} \left( \frac{z_0}{z} \right)^k \right| \leq \frac{d^{p+1}}{(p+1)(1-d)}, \]

where \(d = |R/z|\).

3. Suppose we have an expansion

\[ \phi(z) = a_0 \log(z - z_0) + \sum_{k=1}^{\infty} \frac{a_k}{(z - z_0)^k} \]

that is valid for \(|z - z_0| > R\), for some \(R > 0\). Using a Taylor series with respect to \(z_0\), show we can also write, for \(|z| > R + |z_0|\),

\[ \phi(z) = a_0 \log(z) + \sum_{k=1}^{\infty} \frac{b_k}{z^k}, \]

where

\[ b_k = \left( \sum_{m=1}^{k} a_m z_0^{k-m} \left( \frac{k-1}{m-1} \right) \right) - \frac{a_0 z_0^k}{k}. \]
4. Suppose we have an expansion

\[ \phi(z) = a_0 \log(z - z_0) + \sum_{k=1}^{\infty} \frac{a_k}{(z - z_0)^k} \]

that is valid for \(|z - z_0| > R\), for some \(R > 0\). Suppose further that \(|z_0| > 2R\). Use Taylor series to show we can also write, for \(|z| < R\),

\[ \phi(z) = \sum_{k=0}^{\infty} b_k z^k, \]

where

\[ b_0 = \sum_{m=1}^{\infty} \frac{a_m}{z_0^m} (-1)^m + a_0 \log(-z_0), \]

and

\[ b_k = \left( \frac{1}{z_0^k} \sum_{m=1}^{\infty} \frac{a_m}{z_0^m} \binom{k_m - 1}{m - 1} (-1)^m \right) - \frac{a_0}{k z_0^k}, \]

for \(k \geq 1\).