Math 102 Applied Linear Algebra

What is Linear Algebra about?
Methods for solving systems of linear equations. (Gaussian elimination, matrices,...). A conceptual geometric frame-work in which to think about many problems. (Vector spaces, Orthogonality...). Factorizations of matrices (Diagonalization for most square matrices, Gaussian elimination, Gram-Schmidt orthogonalization,...)

What is this course about?
This is a second course focusing on computational aspects and applications.
Graphs and networks, Fast fourier transform, Difference and differential equations.
Useful factorizations of matrices: Jordan normal form for all square matrices, Singular value decomposition.
We will start with a rapid review of the basic methods to solve systems of linear equations, matrices and the associated geometric subspaces and concepts.

Lecture 1: 1.2 The geometry of Linear systems of equations.
Let us understand the geometric meaning of 2×2 **system of linear equations**:

Ex 1 Find all solutions to the system $\begin{cases} x_1 + x_2 = 3 \\ 2x_1 - x_2 = 0 \end{cases}$

Sol The solutions to each of the equations form a line in the (x_1, x_2)-plane. (x_1, x_2) is therefore a solution to the system if and only if it lies on both these lines. The two lines in the plane intersect at the point $(1, 2)$, which is the only solution

Ex 2 Find all solutions to the system $\begin{cases} x_1 + x_2 = 3 \\ 2x_1 + 2x_2 = 0 \end{cases}$

Sol The lines are parallel and don't intersect. No solutions! The system is **inconsistent**

Ex 3 Find all solutions to the system: $\begin{cases} x_1 + x_2 = 3 \\ 2x_1 + 2x_2 = 6 \end{cases}$

Sol Both equations represent the same line. Every point on the line is a solution!

The geometric interpretation of a 2×2 system as intersection of two lines is called the **row picture** since each row (or equation) represents a line.

There is another geometric interpretation of a 2×2 system called the **column picture**: The two separate equations are really one **vector equation**:

(1)
$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} x_1 + \begin{bmatrix} 1 \\ -1 \end{bmatrix} x_2 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

The problem is to find the right **linear combination** of the **column vectors** that produce the vector on the right side. 1 times the first vector plus 2 times the second produces the vector on the right using the parallelogram law.

Finally, there is yet another interpretation using **matrix multiplication**:

(2)
$$\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

where product of the matrix $\begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$ with the column vector $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ is defined by (1).

Similarly a 3×3 system of linear equations

$$\begin{aligned} x_1 - 2x_2 + x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$

geometrically represents the intersection of 3 planes (the row picture)
or the linear combination of the 3 column vectors (the column picture):

$$\begin{bmatrix} 1 \\ 0 \\ -4 \end{bmatrix} x_1 + \begin{bmatrix} -2 \\ 2 \\ 5 \end{bmatrix} x_2 + \begin{bmatrix} 1 \\ -8 \\ -9 \end{bmatrix} x_3 = \begin{bmatrix} 0 \\ 8 \\ -9 \end{bmatrix}$$

Question When can a 3×3 system be solvable for any right hand side?
When two of the 3 planes are not parallel? This is not sufficient!
When the 3 column vectors do not lie in the same plane.

The higher the order of the system and the dimension is the more difficult it is to visualize the row picture whereas the column picture is the same.

1.3 Gaussian eliminations. We want to analytically solve the systems above.
Two systems are called **equivalent** if they have the same solution set.
Ex 4 The systems (I) and (II) are equivalent:

$$(I): \quad \begin{aligned} x_1 + x_2 &= 3 \\ 2x_1 - x_2 &= 0 \end{aligned}, \qquad \Leftrightarrow \qquad (II): \quad \begin{aligned} x_1 + x_2 &= 3 \\ x_2 &= 2 \end{aligned}$$

In fact, if we subtract 2 times the first equation of (I) from the second equation we get

$$\begin{array}{c} \text{[equation 2]} \\ -2\,\text{[equation 1]} \\ \hline \text{[new equation 2]} \end{array} \qquad \begin{array}{r} 2x_1 - x_2 = 0 \\ -2x_1 - 2x_2 = -6 \\ \hline -3x_2 = -6 \end{array}$$

If we divide both sides by -3 we get the second equation of (II).
System II is in **triangular form**. It can be solved by **back-substitution**:
If we plug $x_2 = 2$ into the first equation we get $2x_1 - 2 = 0$, i.e. $x_1 = 1$.
These operations can be done faster by just on the augmented matrix of the system:

$$\left[\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & -1 & 0 \end{array}\right] \Leftrightarrow \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & -3 & -6 \end{array}\right] (2) - 2(1) \Leftrightarrow \left[\begin{array}{cc|c} 1 & 1 & 3 \\ 0 & 1 & 2 \end{array}\right] (2)/(-3)$$

Ex 5 If we follow the same procedure to solve the system in Ex 2 we get into trouble

$$\begin{array}{c} \text{[equation 2]} \\ -2\,\text{[equation 1]} \\ \hline \text{[new equation 2]} \end{array} \qquad \begin{array}{r} 2x_1 + 2x_2 = 0 \\ -2x_1 - 2x_2 = -6 \\ \hline 0 = -6 \end{array} \quad \text{so we get the system} \quad \begin{aligned} x_1 + x_2 &= 3 \\ 0 &= 6 \end{aligned}$$

which is not true. Hence we analytically found that the system in Ex 2 is inconsistent.

We didn't have time to cover the rest so its homework to read it:.
It is easy to solve a system in non-degenerate triangular form by back-substitution. We therefore want to transform $n \times n$ systems into equivalent triangular systems. Let us recall what basic operations we can do that leads to equivalent systems:
1. A multiple of one equation may be added to another.
2. We can change the order of any two equations
3. Both sides of an equation can be multiplied by the same nonzero number.

We want to solve the 3×3 system

$$\begin{aligned} x_1 - 2x_2 + \ \ x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -4x_1 + 5x_2 + 9x_3 &= -9 \end{aligned}$$

To minimize the writing it is convenient to only write out the **coefficient matrix**:

$$\begin{bmatrix} 1 & -2 & 1 \\ 0 & 2 & -8 \\ -4 & 5 & 9 \end{bmatrix}, \qquad \text{and right hand side column vector} \qquad \begin{bmatrix} 0 \\ 8 \\ -9 \end{bmatrix}$$

or to combine them in one to the **augmented matrix** of the system.

$$\left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{array}\right] \qquad \text{or just} \qquad \begin{bmatrix} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ -4 & 5 & 9 & -9 \end{bmatrix}.$$

Ex 6 Transform the system above into an equivalent triangular system and solve it.
Sol We want to eliminate x_1 from the last equation by using the first:

$$\begin{array}{c} \text{[equation 3]} \\ +4\,\text{[equation 1]} \\ \hline \text{[new equation 3]} \end{array} \qquad \begin{array}{r} -4x_1 + 5x_2 + 9x_3 = -9 \\ 4x_1 - 8x_2 + 4x_3 = 0 \\ \hline -3x_2 + 13x_3 = -9 \end{array}$$

After some practice this calculation is usually performed mentally.
Hence we get the system (written in both ways for comparison)

$$\begin{aligned} x_1 - 2x_2 + \ \ x_3 &= 0 \\ 2x_2 - 8x_3 &= 8 \\ -3x_2 + 13x_3 &= -9 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 2 & -8 & 8 \\ 0 & -3 & 13 & -9 \end{array}\right] \ (3) + 4(1)$$

Now first multiply the second equation by 1/2:

$$\begin{aligned} x_1 - 2x_2 + \ \ x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ -3x_2 + 13x_3 &= -9 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & -3 & 13 & -9 \end{array}\right] \ (2)/2$$

We now want to eliminate x_2 from the last equation by adding 3 times the second:

$$\begin{aligned} x_1 - 2x_2 + \ \ x_3 &= 0 \\ x_2 - 4x_3 &= 4 \\ x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 1 & 0 \\ 0 & 1 & -4 & 4 \\ 0 & 0 & 1 & 3 \end{array}\right] \ (3) + 3(1)$$

Hence we got an equivalent system in non-degenerate triangular form.
Because the diagonal entries are nonvanishing we can solve it using back substitution:

$$\begin{aligned} x_1 - 2x_2 \quad &= -3 \\ x_2 \quad &= 16 \\ x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & -2 & 0 & -3 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{l} (1)-(3) \\ (2)+4(3) \\ \end{array}$$

Now having cleared up the column above x_3 in equation 3, move back to the x_2 in equation 2 and use it to eliminate the $-2x_2$ above it. Adding 2 times the second equation to the first gives

$$\begin{aligned} x_1 \quad &= 29 \\ x_2 \quad &= 16 \\ x_3 &= 3 \end{aligned} \qquad \left[\begin{array}{ccc|c} 1 & 0 & 0 & 29 \\ 0 & 1 & 0 & 16 \\ 0 & 0 & 1 & 3 \end{array}\right] \begin{array}{l} (1)+2(2) \\ \\ \end{array}$$

Operations on the system corresponds to operations on the augmented matrix.
The **Elementary Row Operations** on a matrix are
1. A multiple of one row may be added to another.
2. Interchange two rows
3. Multiplied all entries in a row by the same nonzero number.
Two matrices are **row equivalent** if one can be transformed into the other by elementary row operations. Two systems have the same solution set if their augmented matrices are row equivalent.