Lecture 10: 3.2-3.3 Projections onto lines and planes. (This did not take a whole lecture)

We will now calculate the **orthogonal projection** of $\mathbf{b}$ onto a line through the origin in the direction of $\mathbf{a}$. It is a vector $\mathbf{p} = \alpha\,\mathbf{a}$ in the direction of $\mathbf{a}$, such that $\mathbf{e} = \mathbf{b} - \mathbf{p}$ is orthogonal to $\mathbf{a}$:

$$\mathbf{a}\cdot(\mathbf{b} - \alpha\mathbf{a}) = 0 \quad \Leftrightarrow \quad \mathbf{a}\cdot\mathbf{b} - \alpha\,\mathbf{a}\cdot\mathbf{a} = 0 \quad \Leftrightarrow \quad \alpha = \frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}} \quad \text{so} \quad \mathbf{p} = \mathbf{a}\,\frac{\mathbf{a}\cdot\mathbf{b}}{\mathbf{a}\cdot\mathbf{a}}.$$

The projection $\mathbf{b} \to \mathbf{p}$ is a linear transformation and and this can be written as matrix multiplication $\mathbf{p} = P\mathbf{b}$ with the matrix

$$P = \frac{\mathbf{a}\,\mathbf{a}^T}{\mathbf{a}^T\mathbf{a}}.$$

Here, since $\mathbf{a}$ is a column vector or an $n\times 1$ matrix, $\mathbf{a}\,\mathbf{a}^T$ is an $n\times n$ matrix, whereas $\mathbf{a}^T\mathbf{a}$ is a 1×1 matrix or just a number. P is called a projection matrix, it satisfies $P^T = P$ and $P^2 = P$.

We will now calculate the orthogonal projection of a vector $\mathbf{b}$ onto a plane W through the origin spanned by the two vectors $\mathbf{a}_1$ and $\mathbf{a}_2$. It is the vector in the plane $\mathbf{p} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2$ such that the vector $\mathbf{e} = \mathbf{b} - \mathbf{p}$ is orthogonal to the plane. Since the plane is spanned by $\mathbf{a}_1$ and $\mathbf{a}_2$ this means that

$$\mathbf{a}_1\cdot\big(\mathbf{b} - (x_1\mathbf{a}_1 + x_1\mathbf{a}_2)\big) = 0, \qquad \text{and} \qquad \mathbf{a}_2\cdot\big(\mathbf{b} - (x_1\mathbf{a}_1 + x_1\mathbf{a}_2)\big) = 0.$$

This is a system of two equations that can be solved for the two unknowns x_1 and x_2. However, its better to write this in matrix notation. If $A = [\,\mathbf{a}_1\ \mathbf{a}_2\,]$, then $A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2$, and the above two equations become

$$A^T(\mathbf{b} - A\mathbf{x}) = 0 \quad \Leftrightarrow \quad A^T\mathbf{b} - A^TA\mathbf{x} = 0 \quad \Leftrightarrow \quad \mathbf{x} = (A^TA)^{-1}A^T\mathbf{b},$$

so

$$\mathbf{p} = P\,\mathbf{b}, \qquad \text{where} \qquad P = A(A^TA)^{-1}A^T.$$

If $\mathbf{b}$ is a vector in $\mathbf{R}^3$ then A is a 3×2 matrix and A^TA is a 2×2 matrix.

We claim that if the columns $\mathbf{a}_1$ and $\mathbf{a}_2$ are linearly independent then A^TA is invertible. In fact suppose that $A^TA\mathbf{x} = 0$ and take the dot product of this with $\mathbf{x}$ to get

$$0 = \mathbf{x}^TA^TA\mathbf{x} = (A\mathbf{x})^TA\mathbf{x} = \|A\mathbf{x}\|^2$$

so $A\mathbf{x} = 0$, which implies that $\mathbf{x} = 0$ since the columns of A are independent and hence A^TA is invertible.