Lecture 15: 4.3 Formulas for the determinants.

Let $\omega(\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n)$ be a multilinear antisymmetric n-form on $\mathbf{R}^n$. The **multilinearity** means that it is a linear function of each argument when the other arguments are fixed; for each k

$$(3) \qquad \omega(\mathbf{a}_1, \dots, \mathbf{a}_{k-1}, \alpha\, \mathbf{a}_k + \beta\, \mathbf{b}_k, \mathbf{a}_{k+1}, \dots, \mathbf{a}_n\big) \\ = \alpha\, \omega(\mathbf{a}_1, \dots, \mathbf{a}_{k-1}, \mathbf{a}_k, \mathbf{a}_{k+1}, \dots, \mathbf{a}_n\big) + \beta\, \omega(\mathbf{a}_1, \dots, \mathbf{a}_{k-1}, \mathbf{b}_k, \mathbf{a}_{k+1}, \dots, \mathbf{a}_n\big),$$

and the **antisymmetry** means that for any pairs of indices $j < k$

$$(2) \qquad \omega(\mathbf{a}_1, \dots, \mathbf{a}_{j-1}, \mathbf{a}_j, \mathbf{a}_{j+1}, \dots \mathbf{a}_{k-1}, \mathbf{a}_k, \mathbf{a}_{k+1}, \dots, \mathbf{a}_n\big) \\ = -\,\omega(\mathbf{a}_1, \dots, \mathbf{a}_{j-1}, \mathbf{a}_k, \mathbf{a}_{j+1}, \dots \mathbf{a}_{k-1}, \mathbf{a}_j, \mathbf{a}_{k+1}, \dots, \mathbf{a}_n\big).$$

In particular it follows that $\omega(\mathbf{a}_1, \dots, \mathbf{a}_{j-1}, \mathbf{a}_j, \mathbf{a}_{j+1}, \dots \mathbf{a}_{k-1}, \mathbf{a}_k, \mathbf{a}_{k+1}, \dots, \mathbf{a}_n\big) = 0$ if $\mathbf{a}_j = \mathbf{a}_k$. We further assume that

$$(1) \qquad \omega(\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n) = 1,$$

if $\{\mathbf{e}_1, \mathbf{e}_2, \dots, \mathbf{e}_n\}$ is the standard basis in $\mathbf{R}^n$.

Let us first check that the 2×2 determinant satisfies these properties if $\{\mathbf{a}_1, \dots, \mathbf{a}_n\}$ denotes the row vectors. (1) is obvious from the formula. (2) follows since

$$\begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11}a_{22} - a_{12}a_{21} = -\big(a_{21}a_{12} - a_{11}a_{22}\big) = -\begin{vmatrix} a_{21} & a_{22} \\ a_{11} & a_{12} \end{vmatrix}.$$

In view of (2) we only need to prove (3) for the first row, which follows since

$$\begin{vmatrix} \alpha\, a_{11} + \beta\, b_{11} & \alpha\, a_{12} + \beta\, b_{12} \\ a_{21} & a_{22} \end{vmatrix} = (\alpha\, a_{11} + \beta\, b_{11})a_{22} - (\alpha\, a_{12} + \beta\, b_{12})a_{21} \\ = \alpha\big(a_{11}a_{22} - a_{12}a_{21}\big) + \beta\big(b_{11}a_{22} - b_{12}a_{21}\big) = \alpha \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} + \beta \begin{vmatrix} b_{11} & b_{12} \\ a_{21} & a_{22} \end{vmatrix}.$$

Next we will derive a formula for the $n \times n$ determinant from just using the properties (1), (2) and (3). Let us write $A = [\,\mathbf{a}_1\ \dots\ \mathbf{a}_n\,]$ and expand the columns in the standard basis:

$$\mathbf{a}_j = \sum_{k_j=1}^{n} a_{k_j j}\, \mathbf{e}_{k_j}, \qquad j = 1, \dots, n.$$

Using the linearity we obtain

$$\omega(\mathbf{a}_1, \dots, \mathbf{a}_n) = \sum_{k_1=1}^{n} \cdots \sum_{k_n=1}^{n} a_{k_1 1} \dots a_{k_n n}\, \omega(\mathbf{e}_{k_1}, \dots, \mathbf{e}_{k_n})$$

Because of the antisymmetry (2) $\omega(\mathbf{e}_{k_1}, \dots, \mathbf{e}_{k_n}) = 0$ if two of its arguments are the same so $k_1, \dots, k_n$ all have to be different. Hence $\{k_1, k_2, k_3, \dots, k_n\}$ has to be a permutation of $\{1,2,3,\ldots,n\}$, i.e. a rearrangement of the order. E.g $\{4,3,1,2\}$ and $\{3,4,2,1\}$ are permutations of $\{1,2,3,4\}$. Hence

$$\omega(\mathbf{a}_1,\ldots,\mathbf{a}_n)=\sum_{\sigma\in\mathrm{per}\{1,\ldots,n\}} a_{\sigma(1)1}\ldots a_{\sigma(n)n}\,\omega(\mathbf{e}_{\sigma(1)},\ldots,\mathbf{e}_{\sigma(n)}),$$

where the sum is over all permutations $\{\sigma(1),\ldots,\sigma(n)\}$ of $\{1,\ldots,n\}$. By using (2) to switch the order of the arguments $\mathbf{e}_{\sigma(1)},\ldots,\mathbf{e}_{\sigma(n)}$ in $\omega(\mathbf{e}_{\sigma(1)},\ldots,\mathbf{e}_{\sigma(n)})$ we see

$$\omega(\mathbf{e}_{\sigma(1)},\ldots,\mathbf{e}_{\sigma(n)})=(-1)^{\mathrm{order}(\sigma)}\omega(\mathbf{e}_1,\ldots,\mathbf{e}_n)$$

where the order of the permutation σ is the number of switches required to get $\{\sigma(1),\ldots,\sigma(n)\}$ to its natural order $\{1,\ldots,n\}$. By (1) $\omega(\mathbf{e}_1,\ldots,\mathbf{e}_n)=1$ so

$$\omega(\mathbf{e}_{\sigma(1)},\ldots,\mathbf{e}_{\sigma(n)})=\mathrm{sign}(\sigma)$$

where the sign is $+1$ if it takes an even number of switches to get σ to its natural order and -1 if it takes an odd number of switches. Hence we obtain the formula

$$\omega(\mathbf{a}_1,\ldots,\mathbf{a}_n)=\sum_{\sigma\in\mathrm{per}\{1,\ldots,n\}} a_{\sigma(1)1}\ldots a_{\sigma(n)n}\ \mathrm{sign}(\sigma).$$

The only problem is that the sign of a permutation is not yet well defined, since we have not proven that you can't switch σ to its natural order by both an even and an odd number of switches. Let us therefore define the sign of a permutation in such a way that its clear that it satisfies that it changes sign under a switch and the sign of the identity permutation is one. The sign of a permutation σ is defined according to the number of inversions in σ. An **inversion** is a pair $\sigma(i)>\sigma(j)$ with $i<j$, i.e. it comes in the wrong order, a bigger one before a smaller one. A permutation is called **even or odd** according two whether the number of **inversions** in its result $\{\sigma(1),...,\sigma(n)\}$ is an even or odd integer. The **sign** of a permutation sign(σ) is 1 if the permutation is even and -1 if it is odd. E.g. $\{3,2,5,1,4\}$ have five inversions $(3,2)$, $(3,1)$, $(2,1)$, $(5,1)$ and $(5,4)$ so the sign is -1. It is clear that the sign of the identity permutation 1 since there are no inversion in it. We just have to check that the so defined sign changes with a simple switch. This is clear if we switch two neighbors, since only the orders of the two changes. The desired result will follow if we show that we can do any switch by an odd number of switches of neighbors. We need $\ell-k$ exchanges of neighbors to move an entry in place k to place ℓ. Then $\ell-k-1$ exchanges move the one originally in place ℓ a(and now found in place $\ell-1$) back to place k. Since $\ell-k+(\ell-k-1)$ is odd, the result follows. (Check this in some simple case to see what goes on.)

4.4 Applications of the Determinants-Volume.

Recall that

$$\det(AB) = \det(A)\det(B), \qquad \det(A^T) = \det(A)$$

In particular if Q is an orthogonal matrix, i.e. $Q^T Q = I$ then $\det(Q) = 1$ and

$$\det(QA) = \det(A)$$

Theorem If A is a 2×2 matrix, the area of the parallelogram determined by the columns of A is $|\det A|$. If A is a 3×3 matrix, the area of the parallelogram determined by the columns of A is $|\det A|$.

Pf of the 2×2 cases. The theorem is obviously true for diagonal 2×2 matrices:

$$\left|\det \begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\right| = |ad| = \text{Area of rectangle with sides } a \text{ and } d$$

We will show that any 2×2 matrix $A = [\,\mathbf{a}_1\ \mathbf{a}_2\,]$ can be transformed into a diagonal matrix in a way that changes neither the area of the associated parallelogram nor $|\det A|$. We know that the absolute value of the determinant is unchanged when two columns are interchanged or a multiple of one column is added to another and its easy to see that one can transform A into diagonal form with such operations. Column interchanges do not change the parallelogram at all so it suffices to prove the following fact: The area of the parallelogram determined by $\mathbf{a}_1$ and $\mathbf{a}_2$ equals the area of the parallelogram determined by $\mathbf{a}_1$ and $\mathbf{a}_2 + c\mathbf{a}_1$ for any c. This follows from that the points $\mathbf{a}_2$ and $\mathbf{a}_2 + c\mathbf{a}_1$ have the same perpendicular distance to the line through $\mathbf{0}$ and $\mathbf{a}_1$.

Geometric definition The magnitude of the determinant is the volume of the n-dimensional parallelepiped with the column (or row) vectors as it sides. The sign of the determinant depends on the orientation of the column (or row) vectors.

In 2 and 3 dimensions it was proven in the multi-variable/vector calculus classes that the magnitude of the cross product of two vectors gives the area and the scalar triple product of three vectors gives the volume. The sign of the determinant is positive if the vectors form a positively oriented system.