Lecture 16: 5.1 Eigenvalues and Eigenvectors.

Let A be an $n \times n$ matrix. A vector $\mathbf{x}$ is called an **eigenvector** and λ an **eigenvalue** if

$$A\mathbf{x} = \lambda\mathbf{x},$$

i.e. multiplication with A acts in a very simple way on $\mathbf{x}$. The goal is to find n linearly independent eigenvectors

$$A\mathbf{x}_k = \lambda_k \mathbf{x}_k, \qquad k = 1, \dots, n.$$

If $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ form a basis then any vector $\mathbf{x}$ can be expressed in term of them and so the matrix A is completely determined by its action on the n eigenvectors:

$$A\big(c_1\mathbf{x}_1 + \cdots + c_n\mathbf{x}_n\big) = c_1 A\mathbf{x}_1 + \cdots + c_n A\mathbf{x}_n = c_1\lambda_1\mathbf{x}_1 + \cdots + c_n\lambda_n\mathbf{x}_n.$$

A scalar λ is an eigenvalue if and only if there is $\mathbf{x} \neq \mathbf{0}$ such that

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

The set of all solutions is called the **eigenspace** corresponding to eigenvalue λ. Existence of a nontrivial solution $\mathbf{x}$ is equivalent to that A is not invertible which is equivalent to

$$p(\lambda) \equiv \det(A - \lambda I) = 0.$$

The **characteristic polynomial** $p(\lambda)$ for the matrix A, is a polynomial of degree n, and its roots are exactly the eigenvalues of A. If $p(\lambda)$ has n different real roots then A has n different linearly independent eigenvectors.

If A has n linearly independent eigenvectors then we can solve the system of differential equations

$$\mathbf{x}'(t) = A\mathbf{x}(t), \qquad \mathbf{x}(0) = \mathbf{x}_0$$

easily. In fact the solution is given by

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t}\mathbf{x}_1 + \cdots + c_n e^{\lambda_n t}\mathbf{x}_n$$

since

$$\mathbf{x}'(t) = c_1\lambda_1 e^{\lambda_1 t}\mathbf{x}_1 + \cdots + c_n\lambda_n e^{\lambda_n t}\mathbf{x}_n$$

is equal to

$$A\mathbf{x}(t) = c_1 e^{\lambda_1 t} A\mathbf{x}_1 + \cdots + c_n e^{\lambda_n t} A\mathbf{x}_n = c_1 e^{\lambda_1 t}\lambda_1\mathbf{x}_1 + \cdots + c_n e^{\lambda_n t}\lambda_n\mathbf{x}_n.$$

Ex 1 Suppose that A is the matrix corresponding to the linear transformation which is the reflection in the line $x - y = 0$. Then $A\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x \end{bmatrix}$. The vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is mapped to the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and the vector $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is mapped to the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$, so the matrix is $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. The eigenvalues of A satisfies the equation $\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 1 & -\lambda \end{vmatrix} = \lambda^2 - 1 = 0$, i.e. $\lambda_1 = -1$ and $\lambda_2 = 1$. We will now solve for the eigenvector corresponding to the eigenvalue -1: $(A + I)\mathbf{x}_1 = \mathbf{0}$:

$$\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

i.e. $x + y = 0$. We can therefore take the eigenvector $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$. We will now solve for the eigenvector corresponding to the eigenvalue -1: $(A - I)\mathbf{x}_2 = \mathbf{0}$:

$$\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix},$$

i.e. $-x + y = 0$. We can therefore take the eigenvector $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Note the geometric meaning of the eigenvectors. $\mathbf{x}_2$ is along the line $x - y = 0$ which is not changed by the reflection and $\mathbf{x}_2$ is perpendicular to the line so its reflected to $-\mathbf{x}_2$.

Ex 2 Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$.

Sol This is the matrix for a rotation $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, with an angle $\theta = \pi/4$ and can not have any real eigenvectors unless the rotation a multiple of π.
The eigenvalues are solution of:
$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -1 & -\lambda \end{vmatrix} = \lambda^2 + 1^2 = (\lambda - i)(\lambda + i) = 0$,
i.e. $\lambda_1 = i$, or $\lambda_2 = -i$. The eigenvectors are solutions to:

$$(A - \lambda_1 I)\mathbf{x}_1 = \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \begin{matrix} ix_1 + x_2 = 0 \\ -x_1 + ix_2 = 0 \end{matrix} \quad \Leftrightarrow \quad \mathbf{x}_1 = \alpha \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$$(A - \lambda_2 I)\mathbf{x}_1 = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \begin{matrix} -ix_1 + x_2 = 0 \\ -x_1 - ix_2 = 0 \end{matrix} \quad \Leftrightarrow \quad \mathbf{x}_2 = \beta \begin{bmatrix} i \\ 1 \end{bmatrix}$$

Even though in many applications we are looking for real solutions the complex solutions can still be helpful on the way towards a final answer as we shall see.

Ex 3 Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$.

Sol The eigenvalues are solution of the characteristic equation:
$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ 0 & -\lambda \end{vmatrix} = \lambda^2 = 0$

The eigenvalues are $\lambda_1 = \lambda_2 = 0$. The eigenvectors are solutions to $A\mathbf{x} = \mathbf{0}$:

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad y = 0$$

so the only linearly independent eigenvector is $\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$.

Ex 4 Find the eigenvalues and eigenspaces of $A = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix}$.

Sol Characteristic polynomial $(2 - \lambda)^2(4 - \lambda)$.

$A - 2I = \begin{bmatrix} 2 & 0 & 0 \\ -1 & 3 & 1 \\ -1 & 1 & 3 \end{bmatrix} - \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ -1 & 1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$ and $(A - 2I)\mathbf{x} = \mathbf{0}$ has augmented matrix $\begin{bmatrix} 0 & 0 & 0 & 0 \\ -1 & 1 & 1 & 0 \\ -1 & 1 & 1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \Leftrightarrow \quad x_1 - x_2 - x_3 = 0$

and hence $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_2 + x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ span the eigenspace.

$A - 4I = \begin{bmatrix} -2 & 0 & 0 \\ -1 & -1 & 1 \\ -1 & 1 & 1 \end{bmatrix}$, augmented matrix $\begin{bmatrix} -2 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

$\Leftrightarrow \begin{cases} x_1 = 0, \\ x_2 - x_3 = 0 \end{cases}$ and hence $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$.