Lecture 17: 5.2 Diagonalization.

Discrete Dynamical Systems.

Ex Denote the owl and rat population at time k by O_k and R_k respectively. The rat and owl populations die and reproduce depending on the food supply. We are assuming that there is an unlimited amount of food supply for the rats but that the owls feed on the rats. We assume that the populations at time $k+1$ can be calculated from the populations at time k according to the formulas

$$\begin{aligned} O_{k+1} &= 0.5\, O_k + 0.4\, R_k \\ R_{k+1} &= -p\, O_k + 1.1\, R_k \end{aligned}$$

Let $p = 0.104$. Find the owl and rat populations at time k in terms of the populations at time 0.
Sol. We can write it as a discrete dynamical system:

$$\mathbf{x}_{k+1} = A\mathbf{x}_k, \qquad \text{where} \qquad \mathbf{x}_k = \begin{bmatrix} O_k \\ R_k \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 0.5 & 0.4 \\ -.104 & 1.1 \end{bmatrix}.$$

Then $\mathbf{x}_1 = A\mathbf{x}_0$, $\mathbf{x}_2 = A\mathbf{x}_1 = A^2\mathbf{x}_0$, and in general $\mathbf{x}_k = A^k\mathbf{x}_0$. If we easily could calculate A^k our problem would be solved, but just multiplying these matrices together is a lot of calculations, so we will try to find a better method.

Recall that A acts in a very simpler way on its eigenvectors: $A\mathbf{v}_i = \lambda_i\mathbf{v}_i$ and hence $A^k\mathbf{v}_i = \lambda_i^k\mathbf{v}_i$. The eigenvalues for the matrix A are $\lambda_1 = 1.02$ and $\lambda_2 = 0.58$ and the eigenvectors are $\mathbf{v}_1 = \begin{bmatrix} 10 \\ 13 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$. An initial $\mathbf{x}_0$ can be written $\mathbf{x}_0 = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$. Then for $k \geq 0$

$$\mathbf{x}_k = A^k\mathbf{x}_0 = c_1A^k\mathbf{v}_1 + c_2A^k\mathbf{v}_2 = c_1\lambda_1^k\mathbf{v}_1 + c_2\lambda_2^k\mathbf{v}_2 = c_1(1.02)^k \begin{bmatrix} 10 \\ 13 \end{bmatrix} + c_2(0.58)^k \begin{bmatrix} 5 \\ 1 \end{bmatrix}$$

As k becomes large the first state will dominate and the other will go to $\mathbf{0}$ unless the initial conditions are such that $c_1 = 0$ in which case the whole solution goes to $\mathbf{0}$. We have succeeded in calculating $\mathbf{x}_k$. Could it be that this somehow can be used to also calculate A^k? The answer is yes. Let us examine the formula for $A^k\mathbf{x}_0$ above

$$A^k\mathbf{x}_0 = c_1\lambda_1^k\mathbf{v}_1 + c_2\lambda_2^k\mathbf{v}_2 = [\,\mathbf{v}_1 \quad \mathbf{v}_2\,] \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

It still remains to find $\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ in terms of $\mathbf{x}_0$:

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = [\,\mathbf{v}_1 \quad \mathbf{v}_2\,] \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \mathbf{x}_0$$

Putting things together we have obtained a formula for A^k:

$$A^k\mathbf{x}_0 = [\,\mathbf{v}_1 \quad \mathbf{v}_2\,] \begin{bmatrix} \lambda_1^k & 0 \\ 0 & \lambda_2^k \end{bmatrix} [\,\mathbf{v}_1 \quad \mathbf{v}_2\,]^{-1}\mathbf{x}_0 = [\,\mathbf{v}_1 \quad \mathbf{v}_2\,] \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}^k [\,\mathbf{v}_1 \quad \mathbf{v}_2\,]^{-1}\mathbf{x}_0$$

5.2 Diagonalization.
A square matrix A is called **diagonalizable** if it can be written $A = S\Lambda S^{-1}$, where Λ is diagonal and S is invertible.
When is A diagonalizable and if it is how do we find D and P?
The answer lies in the eigenvalues and eigenvectors.

In general if A is an $n \times n$ matrix with n linearly independent eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ and eigenvalues $\lambda_1, \ldots, \lambda_n$ then

$$A\,[\,\mathbf{v}_1 \ \cdots \ \mathbf{v}_n\,] = [\,A\mathbf{v}_1 \ \cdots \ A\mathbf{v}_n\,] = [\,\lambda_1\mathbf{v}_1 \ \cdots \ \lambda_n\mathbf{v}_n\,] = [\,\mathbf{v}_1 \ \cdots \ \mathbf{v}_n\,]\begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{bmatrix}$$

and hence

$$A = [\,\mathbf{v}_1 \ \cdots \ \mathbf{v}_n\,]\begin{bmatrix}\lambda_1 & & 0\\ & \ddots & \\ 0 & & \lambda_n\end{bmatrix}[\,\mathbf{v}_1 \ \cdots \ \mathbf{v}_n\,]^{-1}$$

Ex If possible diagonalize the reflection matrix $A = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$.
Sol. We previously calculated the eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 1$ and the corresponding eigenvectors $\mathbf{v}_1 = \begin{bmatrix}1\\ -1\end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix}1\\ 1\end{bmatrix}$. Note the geometric meaning of the eigenvectors. $\mathbf{v}_2$ is along the line $x - y = 0$ which is not changed by the reflection and $\mathbf{v}_1$ is perpendicular to the line so its reflected to $-\mathbf{v}_1$. Hence

$$A = S\Lambda S^{-1}$$

where

$$S = \begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}, \qquad \Lambda = \begin{bmatrix}-1 & 0\\ 0 & 1\end{bmatrix}, \qquad S^{-1} = \frac{1}{2}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}.$$

and we used the formula $\begin{bmatrix}a & b\\ c & d\end{bmatrix}^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix}d & -b\\ -c & a\end{bmatrix}$, for the inverse. Hence

$$A^k = S\Lambda^k S^{-1} = \begin{bmatrix}1 & 1\\ -1 & 1\end{bmatrix}\begin{bmatrix}(-1)^k & 0\\ 0 & 1\end{bmatrix}\frac{1}{2}\begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}$$

If k is even the $\Lambda^k = I$ so $A^k = SIS^{-1} = I$ but if k is odd then $\Lambda^k = \Lambda$ so $A^k = A$. This also makes sense geometrically that if we reflect an even number of times we get back to where we started.
Diagonalization Theorem An $n \times n$ matrix is diagonalizable A if and only if it has n linearly independent eigenvectors.

Theorem If $\lambda_1, \ldots, \lambda_n$ are distinct eigenvalues of an $n \times n$ matrix A with corresponding eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$, then $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent.
Proof Let us prove the result for $n = 2$ to show the idea. Suppose that $\mathbf{v}_1$ and $\mathbf{v}_2$ are linearly dependent, i.e. there are c_1 and c_2 not both 0 such that

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 = \mathbf{0}.$$

Then

$$A\big(c_1\mathbf{v}_1 + c_2\mathbf{v}_2\big) = c_1\lambda_1\mathbf{v}_1 + c_2\lambda_2\mathbf{v}_2 = \mathbf{0}.$$

Multiply the first equation by λ_1 and subtracting it from the second gives

$$c_2(\lambda_2 - \lambda_1)\mathbf{v}_2 = \mathbf{0}.$$

Since we assumed that $\lambda_1 \neq \lambda_2$ it follows that $c_2 = 0$ and hence $c_1 = 0$.