Lecture 19: 5.4 Differential Equations and the Exponential matrix.
In this section we consider a system of differential equations

$$\frac{d\mathbf{u}}{dt} = A\mathbf{u}, \quad \mathbf{u}(0) = \mathbf{u}_0 \qquad \text{where} \qquad A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}, \qquad \mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}.$$

How would we solve it? The corresponding system in one dimension

$$\frac{du}{dt} = au, \quad u(0) = u_0$$

has the solution

$$u(t) = a^{at} u_0.$$

We will show that there is a matrix function e^{tA} called the exponential matrix such that the solution to the system is given by

$$\mathbf{u}(t) = e^{tA}\mathbf{u}_0$$

Let λ_k and $\mathbf{x}_k$, be the eigenvalues and eigenvectors of A:

$$A\mathbf{x}_k = \lambda_k \mathbf{x}_k, \qquad k = 1, 2$$

In this case $\lambda_1 = -3$, $\mathbf{x}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$, and $\lambda_2 = -1$, $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Its easy to check that

$$\mathbf{u}(t) = c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2$$

is a solution to the differential equation and this can be written

$$\mathbf{u}(t) = S \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \qquad \text{where} \quad S = [\,\mathbf{x}_1 \ \ \mathbf{x}_2\,]$$

Moreover, since $\mathbf{x}_1$ and $\mathbf{x}_2$ are linearly independent we can find a solution satisfying any initial condition in this form

$$\mathbf{u}(0) = S \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \mathbf{u}_0$$

Hence we have obtained a formula for the solution to the initial value problem

$$\mathbf{u}(t) = S \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} S^{-1}\mathbf{u}_0$$

We could now use this as the definition of the exponential matrix e^{tA}, but instead we will define the exponential matrix using the power series $e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \ldots$, and we define

$$e^{tA} = I + tA + \frac{t^2}{2}A^2 + \frac{t^3}{3!}A^3 + \cdots + \frac{A^n}{n!}A^n + \ldots$$

The terms makes sense so its just a matter of if the series converges. To show that it converges one introduces some matrix norm $\|A\| = \sup_{\|x\|=1} \|Ax\|$ and then the nth term is bounded by $t^2\|A^n\|/n! \le (t\|A\|)^n/n!$, which tends to 0 as $n \to \infty$, since $a^n/n! \to 0$ for any fixed a. In order for $\mathbf{u}(t) = e^{tA}\mathbf{u}_0$ to be a solution of $\mathbf{u}' = A\mathbf{u}$, and $\mathbf{u}(0) = \mathbf{u}_0$ we must have

$$\frac{d}{dt}e^{tA} = Ae^{tA}, \qquad e^{0A} = I$$

The last part easy follows from putting $t = 0$ in the series. The first part follows by differentiating the series

$$\frac{d}{dt}e^{tA} = A+tA^2+\ldots\frac{t^{n-1}}{(n-1)!}A^n+\cdots = A\Big(I+tA+\cdots+\frac{t^{n-1}}{(n-1)!}A^{n-1}+\ldots\Big) = Ae^{tA}$$

In general it is a bit difficult to use the infinite series to calculate the exponential matrix. However, in case A is diagonalizable: $A = S\Lambda S^{-1}$, where

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

it is easy:

$$\begin{aligned} e^{tA} &= I + tS\Lambda S^{-1} + \frac{t^2}{2}S\Lambda S^{-1}S\Lambda S^{-1} + \cdots + \frac{t^n}{n!}S\Lambda S^{-1}\ldots S\Lambda S^{-1} + \ldots \\ &= S\Big(I + t\Lambda + \frac{t^2}{2}\Lambda^2 + \cdots + \frac{t^n}{n!}\Lambda^n + \ldots\Big)S^{-1} \end{aligned}$$

Here

$$\begin{aligned} e^{t\Lambda} &= I + t\Lambda + \frac{t^2}{2}\Lambda^2 + \cdots + \frac{t^n}{n!}\Lambda^n + \ldots \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} t\lambda_1 & 0 \\ 0 & t\lambda_2 \end{bmatrix} + \begin{bmatrix} \frac{t^2\lambda_1^2}{2} & 0 \\ 0 & \frac{t^2\lambda_2^2}{2} \end{bmatrix} + \cdots + \begin{bmatrix} \frac{t^n\lambda_1^2}{n!} & 0 \\ 0 & \frac{t^n\lambda_2^2}{n!} \end{bmatrix} + \ldots \\ &= \begin{bmatrix} 1 + t\lambda_1 + \frac{t^2\lambda_1^2}{2} + \frac{t^n\lambda_1^2}{n!} + \ldots & 0 \\ 0 & 1 + t\lambda_2 + \frac{t^2\lambda_2^2}{2}\frac{t^n\lambda_2^2+\ldots}{n!} \end{bmatrix} = \begin{bmatrix} e^{t\lambda_1} & 0 \\ 0 & e^{t\lambda_2} \end{bmatrix} \end{aligned}$$

Hence if A is diagonalizable $A = S\Lambda S^{-1}$ then

$$e^{tA} = Se^{t\Lambda}S^{-1}, \qquad \text{where} \quad e^{t\Lambda} = \begin{bmatrix} e^{t\lambda_1} & 0 \\ 0 & e^{t\lambda_2} \end{bmatrix}$$

which is the formula we already obtained earlier.

Stability analysis. See section 5.4 in the book.

Diffusion and derivation of the heat equation. See section 5.4 in the book.