Lecture 20 5.4 Complex Eigenvalues.

Ex 1 Find the eigenvalues and eigenvectors of $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$.

Sol The eigenvalues are and eigenvectors are complex:
$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ -1 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 + 1^2 = (1-\lambda-i)(1-\lambda+i) = 0,$
so the matrix can not be diagonalized SAS^{-1} with a real S. It is a rotation with scaling: $A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} = \sqrt{2}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ -1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} = \sqrt{2}\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$, $\theta = \pi/4$.
It can however we be diagonalized with a complex matrix S. With $\lambda_1 = 1+i$, or $\lambda_2 = 1-i$, the eigenvectors are solutions to:

$$(A - \lambda_1 I)\mathbf{x}_1 = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \begin{matrix} -ix + y = 0 \\ -x - iy = 0 \end{matrix} \quad \Leftrightarrow \quad \mathbf{x}_1 = \begin{bmatrix} -i \\ 1 \end{bmatrix}$$

$$(A - \lambda_2 I)\mathbf{x}_1 = \begin{bmatrix} i & 1 \\ -1 & i \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \begin{matrix} ix + y = 0 \\ -x + iy = 0 \end{matrix} \quad \Leftrightarrow \quad \mathbf{x}_2 = \begin{bmatrix} i \\ 1 \end{bmatrix}$$

Even though in many applications we are looking for real solutions the complex solutions can still be helpful on the way towards a final answer as we shall see.

Ex 2 Solve the system of differential equations

$$\frac{d\mathbf{u}}{dt} = A\mathbf{u}, \qquad \mathbf{u}(0) = \mathbf{u}_0 \qquad \text{where} \qquad A = \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}, \qquad \mathbf{u}(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \end{bmatrix}.$$

With $\mathbf{x}_1$ and $\mathbf{x}_2$ the eigenvectors and λ_1 and λ_2 the eigenvalues of A we have still have in the complex case that the general solution is

$$\mathbf{u}(t) = c_1 e^{\lambda_1 t}\mathbf{x}_1 + c_2 e^{\lambda_2 t}\mathbf{x}_2 = c_1 e^{t+it}\begin{bmatrix} -i \\ 1 \end{bmatrix} + c_2 e^{t-it}\begin{bmatrix} i \\ 1 \end{bmatrix}$$

In fact, if we define e^z for complex $z = a + ib$ either by using the power series or by $e^{a+ib} = e^a e^{ib}$, where $e^{ib} = \cos b + i \sin b$, it is easy to show that

$$\frac{d}{dt} e^{zt} = z e^{zt}$$

We can then rewrite

$$\mathbf{u}(t) = c_1 e^{t+it}\begin{bmatrix} -i \\ 1 \end{bmatrix} + c_2 e^{t-it}\begin{bmatrix} i \\ 1 \end{bmatrix} = e^t\Big(c_1(\cos t + i \sin t)\begin{bmatrix} -i \\ 1 \end{bmatrix} + c_2(\cos t - i \sin t)\begin{bmatrix} i \\ 1 \end{bmatrix}\Big)$$
$$= e^t\Big((c_1 + c_2)\begin{bmatrix} \sin t \\ \cos t \end{bmatrix} + i\,(c_2 - c_1)\begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}\Big) = e^t\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}\begin{bmatrix} d_1 \\ d_2 \end{bmatrix}$$

where $d_2 = c_1 - c_2 = d_2$ and $d_1 = i(c_2 - c_1) = d_1$. By choosing d_1 and d_2 real we get a real solution. We conclude that

$$e^{At} = e^t\begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

5.5 Complex matrices. As we have seen it is natural to consider complex vectors. Recall that for complex numbers $z = a + ib$ we have $|z|^2 = a^2 + b^2 = \overline{z}z$, where the complex conjugate is $\overline{z} = a - ib$. The length of a vector with complex components is $\|\mathbf{z}\|^2 = |z_1|^2 + \cdots + |z_n|^2 = \overline{z}_1 z_1 + \cdots + \overline{z}_n z_n = \overline{\mathbf{z}}^T\mathbf{z}$ where the complex inner product is

$$\overline{\mathbf{x}}^T\mathbf{y} = \overline{x}_1 y_1 + \cdots + \overline{x}_n y_n.$$

We want to take the transpose and conjugate of matrices so we introduce a notation:

$$A^H = \overline{A}^T$$

(In some texts it is called A^*) As for the transpose $(AB)^H = B^H A^H$ and $(A^H)^H = A$. With this notation the inner product is $\mathbf{x}^H\mathbf{y}$ and $\mathbf{x}$ ad $\mathbf{y}$ are called orthogonal if $\mathbf{x}^T\mathbf{y} = 0$. There is an analog of symmetric matrices called **Hermitian** matrices

$$A^H = A.$$

A symmetric real matrix is in particular Hermitian.

Theorem $\mathbf{x}^H A\mathbf{x}$ is real also for complex $\mathbf{x}$ if $A^H = A$.
Proof $(\mathbf{x}^H A\mathbf{x})^H = \mathbf{x}^H A^H (\mathbf{x}^H)^H = \mathbf{a}^H A\mathbf{x}$. This shows that number $\mathbf{x}^H A\mathbf{x}$ is its own conjugate so it must be real.

Theorem Eigenvalues of A are real if $A^H = A$.
Proof Multiply $A\mathbf{x} = \lambda\mathbf{x}$ by $\mathbf{x}^H$; $\mathbf{x}^H A\mathbf{x} = \lambda\mathbf{x}^H\mathbf{x}$. Since the left is real by the previous theorem and $\mathbf{x}^H\mathbf{x}$ is real it follows that λ must be real.

Theorem Eigenvectors for different eigenvalues are orthogonal if $A^H = A$.
Proof If $A\mathbf{x}_1 = \lambda_1\mathbf{x}_1$ and $A\mathbf{x}_2 = \lambda_2\mathbf{x}_2$ then sine $A^H = A$ and λ_1 is real

$$\lambda_1\mathbf{x}_1^H\mathbf{x}_2 = (A\mathbf{x}_1)^H\mathbf{x}_2 = \mathbf{x}_1^H A\mathbf{x}_2 = \lambda_2\mathbf{x}_1^H\mathbf{x}_2$$

It therefore follows that $\mathbf{x}_1^T\mathbf{x}_2 = 0$ since we assumed that $\lambda_1 \neq \lambda_2$.

Spectral Theorem If A is real symmetric with different eigenvalues then it can be factorized $A = Q\Lambda Q^T$, where Q is orthogonal and Λ is diagonal.
Proof We previously showed that we can write $A = S\Lambda S^{-1}$, where the columns of S are the eigenvectors. Since the eigenvectors are orthogonal we can normalize them so they are orthonormal and then $Q = S$ is an orthogonal matrix and $Q^T = S^{-1}$.
Remark The theorem holds even if the eigenvalues are repeated as we will show. The theorem shows that A can be written as a sum of projections:

$$A = Q\Lambda A^T = [\,\mathbf{x}_1 \ \ \ldots \ \ \mathbf{x}_n\,]\,\Lambda\,[\,\mathbf{x}_1 \ \ \ldots \ \ \mathbf{x}_n\,]^T = \lambda_1\mathbf{x}_1\mathbf{x}_1^T + \cdots + \lambda_n\mathbf{x}_n\mathbf{x}_n^T.$$

The natural generalization of orthogonal matrices are **unitary matrices**

$$U^H U = I.$$

Theorem $\|U\mathbf{x}\| = \|\mathbf{x}\|$
Proof $\|U\mathbf{x}\|^2 = (U\mathbf{x})^H U\mathbf{x} = \mathbf{x}^H U^H U\mathbf{x} = \mathbf{x}^T\mathbf{x} = \|\mathbf{x}\|^2$.

Theorem Every eigenvalue of U has absolute value $|\lambda| = 1$.
Proof If $U\mathbf{x} = \lambda\mathbf{x}$ then $\|U\mathbf{x}\| = |\lambda|\|\mathbf{x}\|$ and $\|U\mathbf{x}\| = \|\mathbf{x}\|$.

Theorem Eigenvectors corresponding to different eigenvalues are orthogonal.
Proof If $U\mathbf{x}_i = \lambda_i\mathbf{x}_i$ then $\overline{\lambda}_1\lambda_2\mathbf{x}_1^T\mathbf{x}_2 = (U\mathbf{x}_1)^H U\mathbf{x}_2 = \mathbf{x}_1^H U^H U\mathbf{x}_2 = \mathbf{x}_1^T\mathbf{x}_2$.