Lecture 22: 5.6 and Appendix B: Jordan Normal form.

Theorem (Jordan Normal Form) If A has s independent eigenvectors (and no more), it is similar to a matrix with s **Jordan blocks**;

$$J = M^{-1}AM = \begin{bmatrix} J_1 & & & & \\ & \cdot & & & \\ & & \cdot & & \\ & & & \cdot & \\ & & & & J_s \end{bmatrix}, \qquad \text{where} \qquad J_i = \begin{bmatrix} \lambda_i & 1 & & & \\ & \lambda_i & 1 & & \\ & & \cdot & \cdot & \\ & & & \cdot & 1 \\ & & & & \lambda_i \end{bmatrix}.$$

An example of a matrix in Jordan normal form is

$$J = \begin{bmatrix} 8 & 1 & & & \\ 0 & 8 & & & \\ & & 0 & 1 & \\ & & 0 & 0 & \\ & & & & 0 \end{bmatrix}$$

To understand how you can obtain the Jordan normal form let us first start with the example above and see which matrices can have it as its Jordan normal form. If $M = [\,\mathbf{u}_1\,\mathbf{u}_2\,\mathbf{u}_3\,\mathbf{u}_4\,\mathbf{u}_5\,]$ then

$$A \begin{bmatrix} \\ \mathbf{u}_1\,\mathbf{u}_2\,\mathbf{u}_3\,\mathbf{u}_4\,\mathbf{u}_5 \\ \\ \end{bmatrix} = \begin{bmatrix} \\ \mathbf{u}_1\,\mathbf{u}_2\,\mathbf{u}_3\,\mathbf{u}_4\,\mathbf{u}_5 \\ \\ \end{bmatrix} \begin{bmatrix} 8 & 1 & & & \\ 0 & 8 & & & \\ & & 0 & 1 & \\ & & 0 & 0 & \\ & & & & 0 \end{bmatrix}$$

so $A\mathbf{u}_1 = 8\mathbf{u}_1, \quad A\mathbf{u}_2 = 8\mathbf{u}_2 + \mathbf{u}_1, \quad A\mathbf{u}_3 = \mathbf{0}, \quad A\mathbf{u}_4 = \mathbf{u}_3, \quad A\mathbf{u}_5 = \mathbf{0}.$
Here $\mathbf{u}_1, \mathbf{u}_3, \mathbf{u}_5$ are eigenvectors with eigenvalues, 8, 0 and 0 respectively, and $\mathbf{u}_2, \mathbf{u}_4$ are called generalized eigenvectors, they satisfy $(A - 8I)^2\mathbf{u}_2 = 0$ and $A^2\mathbf{u}_4 = 0$.

Let us now start with a matrix and see the procedure for obtaining the Jordan form. The proof is by induction assuming that we know how to obtain the Jordan form for smaller matrices. We will assume that 0 is an eigenvalue. If it is not we reduce to this case by subtracting off λI for and eigenvalue λ. Now consider

$$A = \begin{bmatrix} 8 & 0 & 0 & 8 & 8 \\ 0 & 0 & 0 & 8 & 8 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 8 \end{bmatrix}$$

Step 1 The column space has dimension $r = 3$ and is spanned by the vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_5$. Within this smaller space we use the induction to find the eigenvectors and generalized eigenvectors $\mathbf{w}_1 = \begin{bmatrix} 8 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{w}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{w}_1 = \begin{bmatrix} 0 \\ 8 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, satisfying

$$A\mathbf{w}_1 = 8\mathbf{w}_1, \quad A\mathbf{w}_2 = 8\mathbf{w}_2 + \mathbf{w}_1, \quad A\mathbf{w}_3 = \mathbf{0}.$$

Step 2 The nullspace contains $\mathbf{e}_2$ and $\mathbf{e}_3$ so its intersection with the column space is spanned by $\mathbf{e}_2$ (or $\mathbf{w}_3$) and hence has dimension $p = 1$. Since it is in the column space we can find a solution to $\mathbf{w}_3 = A\mathbf{w}_4$, $\mathbf{w}_4 = \mathbf{e}_4 - \mathbf{e}_1$.
Step 3 $\mathbf{w}_5 = \mathbf{e}_3$ is in the nullspace but outside the column space satisfying $A\mathbf{w}_3 = \mathbf{0}$.