Lecture 23: 6.1 Min-Max and 6.2 Positive definite matrices.

Min-Max.
A function $F(x, y)$ has a **stationary or critical point** at (x_0, y_0) if the first order derivatives vanishes $F_x(x_0, y_0) = F_y(x_0, y_0) = 0$. The question we want to ask is if the critical point is a local maximum or minimum or neither. For simplicity of notation we will assume that the critical point is $(0, 0)$. By Taylor's formula

$$F(x, y) = F(0, 0) + F_x(0, 0)x + F_y(0, 0)y + P_2(x, y) + R_2(x, y),$$

where

$$P_2(x, y) = \frac{1}{2}\big(F_{xx}(0, 0)x^2 + 2F_{xy}(0, 0)xy + F_{yy}(0, 0)y^2\big)$$

and the error is smaller when (x, y) is close to $(0, 0)$.

$$|R_2(x, y)| \le C(|x| + |y|)^3$$

The behavior of F close to $(0, 0)$ is determined by the behavior of P_2 so we want to study for which constants a, b, c a polynomial

$$P(x, y) = ax^2 + 2bxy + cy^2$$

has a max/min or saddle point. The typical examples are $P = x^2 + y^2$ a max, $P = -(x^2 + y^2)$ a min and $P(x, y) = \pm(x^2 - y^2)$ or $P = 2xy$ a saddle point.
$P(x, y)$ is said to be **positive definite** if $P(x, y) > 0$ for $(x, y) \ne (0, 0)$.
What conditions on a, b, c are needed for $P(x, y)$ to be positive definite?
We must have $P(1, 0) = a > 0$ and $P(0, 1) = c > 0$ but that is not all.
If we complete the square we get

$$P(x, y) = a\Big(x + \frac{b}{a}y\Big)^2 + \Big(c - \frac{b^2}{a}\Big)y^2,$$

so we must have $P\Big(-\frac{b}{a}, 1\Big) = c - \frac{b^2}{a} > 0$ so we must also have that $ac > b^2$. On the other hand it follows that $P(x, y)$ is positive definite if these conditions are true. Now $P(x, y)$ can be written as a bilinear form using a symmetric matrix:

$$P(x, y) = \begin{bmatrix} x \\ y \end{bmatrix}^T \begin{bmatrix} a & b \\ b & c \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \mathbf{x}^T A \mathbf{x}$$

so the condition for positivity is that the determinant $ac - b^2 > 0$ and $a > 0$. An equivalent condition is that the eigenvalues λ_1 and λ_2 are positive since $\lambda_1\lambda_2 = ac - b^2$ and $\lambda_1 + \lambda_2 = a + c$.

Positive definiteness. If A is a symmetric $n \times n$ matrix then the quadratic form

$$\mathbf{x}^T A \mathbf{x} = \sum_{i,j=1}^{n} a_{ij} x_i x_j$$

is said to be positive definite if $\mathbf{x}^T A \mathbf{x} > 0$ for $\mathbf{x} \neq 0$.
This is equivalent to that all the eigenvalues $\lambda_i > 0$.
It is also equivalent to that all the pivots (without row exchange) $d_i > 0$.
In fact A can be diagonalized $A = Q \Lambda Q^T$, $\Lambda = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$ are the eigenvalues so if we set $\mathbf{y} = Q^T \mathbf{x}$ we obtain

$$\mathbf{x}^T A \mathbf{x} = \mathbf{x}^T Q \Lambda Q^T \mathbf{x} = (\mathbf{x} Q^T)^T \Lambda Q \mathbf{x} = \mathbf{y}^T \Lambda \mathbf{y} = \lambda_1 y_1^2 + \cdots + \lambda_n y_n^2$$

which is always positive if the eigenvalues are positive. Similarly, the LU factorization for A becomes $A = LDL^T$ where $D = \mathrm{diag}(d_1, \dots, d_n)$ are the pivots so with $\mathbf{y} = L^T \mathbf{x}$ we obtain

$$\mathbf{x}^T A \mathbf{x} = \mathbf{y}^T D \mathbf{y} = d_1 y_1^2 + \cdots + d_n y_n^2$$

That he LDU factorization becomes $A = LDL^T$ for symmetric matrices was proven in section 1.6.