Lecture 24: 6.3 Singular Value Decomposition.
Any $m \times n$ matrix A can be factored into

$$A = U\Sigma V^T$$

where U is an orthogonal $m \times m$ matrix, V is an orthogonal $n \times n$ matrix and Σ is a diagonal $m \times n$ matrix. The columns of U are eigenvectors of A^TA and the columns of V are eigenvectors of A^TA. The r singular values on the diagonal of Σ are the square roots of the nonzero eigenvalues of AA^T and A^TA.

Proof Let r be the rank of A which is also the rank of A^TA. Since A^TA is symmetric it has a complete set of orthonormal eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$, and suppose they are numbered so the first r corresponds to the nonzero eigenvalues $\sigma_1^2, \dots, \sigma_r^2$. Set $V = [\,\mathbf{v}_1 \dots \mathbf{v}_n\,]$. Note that $AA^T(A\mathbf{v}_i) = A(A^TA\mathbf{v}_i) = \sigma_i^2 A\mathbf{v}_i$, for $i = 1, \dots, r$. Let $\mathbf{u}_i$ be equal to $A\mathbf{v}_i$ normalized, for $i = 1, \dots, r$, and let $\mathbf{u}_i$, for $i = r+1, \dots, m$ be other eigenvectors so that $\mathbf{u}_1, \dots, \mathbf{u}_m$ form an orthonormal basis and set $U = [\,\mathbf{u}_1 \dots \mathbf{u}_n\,]$.

Ex Find a singular value decomposition for $A = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & -1 \end{bmatrix}$.

Sol The matrix $A^TA = \begin{bmatrix} 2 & 2 & -2 \\ 2 & 2 & -2 \\ -2 & -2 & 2 \end{bmatrix}$ has rank 2 so it must have a two dimensional nullspace and hence 0 is a double ewigenvalue. It the easy to find that the eigenvalues are $\lambda_1 = 6$, $\lambda_2 = \lambda_3 = 0$. The corresponding orthonormal eigenvectors can be found $\mathbf{v}_1 = \frac{1}{\sqrt{3}}\begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$. Set $V = [\,\mathbf{v}_1\mathbf{v}_2\mathbf{v}_3\,]$. The last two are found pick two vectors satisfying $x+y-2z = 0$ and applying Gram-Schmidt to get an orthonormal basis. $\Sigma = \begin{bmatrix} \sigma_1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. $\mathbf{u}_1 = A\mathbf{v}_1/\sqrt{6} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Let $\mathbf{u}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -11 \end{bmatrix}$ be a vector orthogonal to $\mathbf{u}_1$ and set $U = [\,\mathbf{u}_1\ \mathbf{u}_2\,]$.

Applications of SVD.

One can approximate large matrices by ones of smaller rank by setting the small singular values to 0.

One can construct a **pseudoinverse** $A^+ = V\Sigma^+U^T$, where Σ^+ is obtained from Σ by inverting the nonzero singular values in the diagonal.