Lecture 3: 1.6 The inverse of a matrix.

The inverse of a real number $a \neq 0$ is a number denoted a^{-1} such that

$$a \cdot a^{-1} = 1$$

An $n{\times}n$ matrix A is said to be **invertible** if there is an $n{\times}n$ matrix A^{-1} such that

$$A^{-1}A = AA^{-1} = I \tag{2.2.1}$$

where I is the identity matrix. The matrix A^{-1} called the **inverse** of A is unique. In fact if $BA = AB = I$ then $B = BI = B(AA^{-1}) = (BA)A^{-1} = IA^{-1} = A^{-1}$. Not all $n \times n$ matrices are invertible. A matrix which is not invertible is called **singular**. An invertible matrix is called **nonsingular**.

Th If A is invertible then the equations $A\mathbf{x} = \mathbf{b}$ has the unique solution $\mathbf{x} = A^{-1}\mathbf{b}$.
Pf If A is invertible then $A^{-1}\mathbf{b}$ is a solution to the system $A\mathbf{x} = \mathbf{b}$. In fact,

$$A(A^{-1}\mathbf{b}) = (AA^{-1})\mathbf{b} = I\,\mathbf{b} = \mathbf{b}.$$

To see that it is the only solution to $A\mathbf{x} = \mathbf{b}$ we multiply both sides by A^{-1} to get

$$A^{-1}A\mathbf{x} = A^{-1}\mathbf{b},$$

and since

$$A^{-1}A\mathbf{x} = I\mathbf{x} = \mathbf{x},$$

it follows that

$$\mathbf{x} = A^{-1}\mathbf{b}.$$

Th Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. If $ad-bc \neq 0$ then A is invertible and $A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$.
If $ad - bc = 0$ then A is not invertible.
Pf If $ad - bc \neq 0$ its easy to check that $AA^{-1} = A^{-1}A = I$:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}.$$

If $ad - bc = 0$ then (a, b) and (c, d) are proportional and the system

$$\begin{aligned} ax_1 + bx_2 &= b_1 \\ cx_1 + dx_2 &= b_2 \end{aligned}$$

does not have a unique solution so the conclusion in the preceding theorem is wrong and hence the assumption must be wrong, i.e. A is not invertible.
Ex Solve the system

$$\begin{aligned} -7x_1 + 3x_2 &= 2 \\ 5x_1 - 2x_2 &= 1 \end{aligned}$$

$$A = \begin{bmatrix} -7 & 3 \\ 5 & -2 \end{bmatrix},\ A^{-1} = \frac{1}{7{\cdot}2 - 3{\cdot}5}\begin{bmatrix} -2 & -3 \\ -5 & -7 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} \text{ so } \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 7 \\ 17 \end{bmatrix}$$

An **inverse** of a transformation $\mathbf{x} \to T(\mathbf{x})$ is a transformation which takes you back $T(\mathbf{x}) \to \mathbf{x}$. The condition $A^{-1}A = I$ says that the inverse of the linear transformation $\mathbf{x} \to A\mathbf{x}$ is the linear transformation $\mathbf{y} \to A^{-1}\mathbf{y}$. In fact, if we compose $\mathbf{x} \to A\mathbf{x}$ with $\mathbf{y} \to A^{-1}\mathbf{y}$ we get $\mathbf{x} \to A\mathbf{x} \to A^{-1}(A\mathbf{x}) = (AA^{-1})\mathbf{x} = I\mathbf{x} = \mathbf{x}$.
Question What is the inverse of a scaling by a factor 3 and what is its matrix? What is the inverse of a rotation counterclockwise angle $\pi/2$ and what is its matrix?

$(AB)^{-1} = B^{-1}A^{-1}$, $(A^{-1})^{-1} = A$ as is clear from the composition of transformations.

Question: How can we calculate the inverse of a matrix?
If we can solve $A\mathbf{x} = \mathbf{y}$ for any $\mathbf{y}$ we will get the inverse $\mathbf{x} = A^{-1}\mathbf{y}$.

Ex Find the inverse of $A = \begin{bmatrix} 1 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

Sol We perform row operations to solve the system $A\mathbf{x} = \mathbf{y}$:

$$\begin{cases} x_1 = y_1 \\ -3x_1 + x_3 = y_2 \\ x_2 = y_3 \end{cases} \Leftrightarrow \begin{cases} x_1 = y_1 \\ x_3 = 3y_1 + y_2 \\ x_2 = y_3 \end{cases} \Leftrightarrow \begin{cases} x_1 = y_1 \\ x_2 = y_3 \\ x_3 = 3y_1 + y_2 \end{cases}$$

adding three times the first equation to the second and then switching the second and the third equations. The system on the right is $\mathbf{x} = A^{-1}\mathbf{y}$ so we must have that $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 3 & 1 & 0 \end{bmatrix}$. Its easy to check that $AA^{-1} = I$.
The calculations above can be performed without writing out the variables as row operations directly to the augmented matrix $[\,A \quad I\,]$;

$$\begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \sim (2)+3(1) \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 3 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \end{bmatrix} \sim \begin{matrix} \\ (3) \\ (2) \end{matrix} \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 3 & 1 & 0 \end{bmatrix}$$

We have found an algorithm for determining if A is invertible and finding the inverse: Calculate the reduced row echelon form of the augmented matrix $[\,A \quad I\,]$. If it is of the form $[\,I \quad B\,]$ then A is invertible and $A^{-1} = B$. Otherwise A is not invertible.

One can also prove that this works multiplying by **elementary** matrices which correspond to elementary row operations. Let $E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$
Multiplying by E_1 adds 3 times row one to row two:
$E_1 A = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -3 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$
Multiplying by E_2 switches row two and row three:
$E_2(E_1 A) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$ Hence

$$E_2 E_1 A = I$$

and multiplying both sides by A^{-1} to the right gives since $AA^{-1} = I$ and $IA^{-1} = A^{-1}$:

$$E_2 E_1 I = A^{-1}$$

Hence a sequence of elementary row operations that reduce A to I reduce I to A^{-1}. This argument assumed that A was invertible, but it also follows since each elementary matrix is invertible since the row operations are reversible and hence

multiplying by the inverse of the elementary matrices gives $A = E_1^{-1}E_2^{-1}$ so A is invertible since its a product of invertible matrices.
The transpose A^T is the matrix with rows and columns interchanged, $(A^T)_{ij} = (A)_{ji}$

Ex If $A = \begin{bmatrix} 1 & 2 & 3 \\ -2 & 0 & -1 \\ 4 & 5 & 2 \end{bmatrix}$ then $A^T = \begin{bmatrix} 1 & -2 & 4 \\ 2 & 0 & 5 \\ 3 & -1 & 2 \end{bmatrix}$.
We have e.g. $(AB)^T = B^T A^T$, $(A+B)^T = A^T + B^T$.

1.7 Special matrices and applications. Suppose that we numerically want to solve a differential equation approximately, i.e. give f we want to find u satisfying

$$-\frac{d^2u}{dx^2} = f(x), \quad 0 \le x \le 1, \qquad u(0) = 0, \quad u(1) = 0.$$

This can e.g. describe the temperature distribution in a rod with the endpoints fixed at 0 degrees and with a heat source $f(x)$. Suppose we only can measure $f(x)$ at a finite set of points $x = h, 2h, \cdots, nh$, where $h = 1/(n+1)$. Let $f_j = f(jh)$. We want to compute approximate values for u at these points $u_1, \cdots, u_n$, the boundary values are $u_0 = u_{n+1} = 0$.

How do we approximate d^2u/dx^2?

$$\frac{\triangle u}{\triangle x}(x) = \frac{u(x+k) - u(x-k)}{2k}$$

How about the second derivative

$$\frac{\triangle^2 u}{\triangle x^2}(x) = \frac{1}{2\ell}\Big(\frac{\triangle u}{\triangle x}(x+\ell)) - \frac{\triangle u}{\triangle x}(x-\ell)\Big)$$

or if we plug in the first expression

$$\frac{\triangle^2 u}{\triangle x^2}(x) = \frac{1}{2\ell}\Big(\frac{u(x+\ell+k) - u(x+\ell-k)}{2k} - \frac{u(x-\ell+k) - u(x-\ell-k)}{2k}\Big)$$

If we choose $\ell = k = h/2$ we get

$$\frac{\triangle^2 u}{\triangle x^2}(x) = \frac{1}{h}\Big(\frac{u(x+h) - u(x)}{h} - \frac{u(x) - u(x-h)}{h}\Big) = \frac{u(x+h) + u(x-h) - 2u(x)}{h^2}$$

Plugging this approximation into the differential equation we get if we multiply by h^2 we get a difference equation:

$$-u_{j+1} + 2u_j - u_{j-1} = h^2 f_j, \qquad 1 \le j \le n, \quad u_0 = u_{n+1} = 0.$$

For $n = 5$ this can be written as a matrix equation

$$\begin{bmatrix} 2 & -1 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 \\ 0 & 0 & -1 & 2 & -1 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \end{bmatrix} = h^2 \begin{bmatrix} f_1 \\ f_2 \\ f_3 \\ f_4 \\ f_5 \end{bmatrix}$$

This is a band matrix an elimination can be done in a few steps.