Lecture 4: 2.1 Vector Spaces. Recall that if $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$ are vectors in Euclidean space we defined the **addition** $\mathbf{x} + \mathbf{y} \in \mathbf{R}^n$ and **scalar multiplication** $\lambda\mathbf{x} \in \mathbf{R}^n$. The addition and scalar multiplication in $\mathbf{R}^n$ satisfy certain properties listed below. There are many other spaces with an addition and scalar multiplication satisfying these properties and these properties are what is used to prove the theorems. Rather then repeating the proofs in each new situation it is more efficient to introduce the concept of an abstract vector space to be a set with addition and scalar multiplication satisfying these properties and once and for all prove the theorems under these assumptions only.

A set V with two operations, addition and multiplication by scalars, defined on it is called a **vector space** if the following properties hold for any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ $\alpha, \beta \in \mathbf{R}$:

1. If $\mathbf{u}, \mathbf{v} \in V$ then $\mathbf{u} + \mathbf{v} \in V$. (closure under addition)
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ (commutative)
3. $(\mathbf{u} + \mathbf{v}) + \mathbf{w}) = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ (associative)
4. There is an element $\mathbf{0} \in V$ such that $\mathbf{u} + \mathbf{0} = \mathbf{u}$ all $\mathbf{u} \in V$ (additive unit)
5. For each $\mathbf{u} \in V$ there is $-\mathbf{u} \in V$ such that $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ (additive inverse)
6. If $\mathbf{u} \in V$ and α is a scalar then $\alpha\,\mathbf{u} \in V$. (closure under scalar multiplication)
7. $\alpha(\mathbf{u} + \mathbf{v}) = \alpha\mathbf{u} + \alpha\mathbf{v}$ (distributive)
8. $(\alpha + \beta)\mathbf{u} = \alpha\mathbf{u} + \beta\mathbf{u}$ (distributive)
9. $(\alpha\beta)\mathbf{u} = \alpha(\beta\mathbf{u})$ (associative)
10. $1 \cdot \mathbf{u} = \mathbf{u}$ (multiplicative unit)

It follows from (1)-(10) that $0\mathbf{u} = \mathbf{0}$, $(-1)\mathbf{u} = -\mathbf{u}$, $c\,\mathbf{0} = \mathbf{0}$.

Linear space would perhaps be a better name. In brief its a set with two operations, addition and scalar multiplication, that allows us to form linear combinations.
It is difficult to understand from the axioms what a vector space is. Instead one has to get a feeling for what it looks like by examples. If you describe to an alien that a chair is something with a seat and a back they will not understand but if you show them many chairs and how they are used they will get a good idea.

$\mathbf{R}^n$ satisfy these properties, and more generally, so do $m \times n$ matrices; $\mathbf{R}^{m\times n}$, with the sum $A{+}B \in \mathbf{R}^{m\times n}$ and scalar multiplication $\lambda A \in \mathbf{R}^{m\times n}$ we previously defined.

Let $I = [0, 1]$ and let $C(I, \mathbf{R})$ be the set of real valued continuous functions $I \to \mathbf{R}$. If f, g are functions and λ a scalar then we can define the functions $f + g$ and λf by $(f + g)(t) = f(t) + g(t)$ and $(\lambda f)(t) = \lambda f(t)$.
All the 10 properties above are satisfied which makes $C(I, \mathbf{R})$ into a vector space. A vector in $\mathbf{R}^n$ is determined by its n components but to specify a function on I we have to give its value at infinitely many points. Still the analogy with $\mathbf{R}^n$ has proved enormously useful. E.g. to find the polynomial that best approximate a function one projects onto the closest polynomial in a certain distance.

The idea of an abstract vector space goes back to Grassmann in 1844. He realized that once geometry is put into this axiomatic algebraic form it would no longer be limited to three dimensional space. However, the contemporary mathematicians failed to recognize the importance of his work and it was not understood until Peano in 1888 published a clear condensed interpretation. He didn't get a university position and during his life he got more recognition for his study of languages.

Ex $F = \{\mathbf{x} \in \mathbf{R}^3;\, x_1 + x_2 + x_3 = 0\}$ with the addition and multiplication coming from $\mathbf{R}^3$ is a vector space. It is some work to check all the 10 axioms but we will see shortly that since its a subset of a vector space we only have to check the closure properties, i.e. if we add two vectors in F and multiply a vector in F by a scalar then the result stays in F, and it follows that all the axioms hold.
Ex $G = \{\mathbf{x} \in \mathbf{R}^3;\, x_1 + x_2 + x_3 = 1\}$ with the addition and multiplication coming from $\mathbf{R}^3$ is not a vector space. In fact, it is easy to see that if we add two vectors $\mathbf{x}, \mathbf{y} \in G$ then $\mathbf{z} = \mathbf{x} + \mathbf{y} \notin G$ since $z_1 + z_2 + z_3 = x_1 + y_1 + x_2 + y_2 + x_3 + y_3 = x_1 + x_2 + x_3 + y_1 + y_2 + y_3 = 1 + 1 = 2 \neq 1$.

Question: When is a subset of a vector space a vector space itself?
We defined a vector spaces V as a set with addition and scalar multiplication that satisfy 10 axioms. However, often we have a subset of a vector space in which case we only need to check that its closed under addition and scalar multiplication:
A subset S of a vector space V is called **subspace** if
(a) $\mathbf{0} \in S$.
(b) $\mathbf{u} + \mathbf{v} \in S$, whenever $\mathbf{u}, \mathbf{v} \in S$.
(c) $\alpha\,\mathbf{u} \in S$, whenever $\mathbf{u} \in S$ and α is a scalar.
A subspace is automatically a vector space in its own right, i.e. with addition and scalar multiplication inherited from (coming from) V it satisfies all the 10 axioms. In fact (1) is (b) and (6) is (c). (2)-(4) and (7)-(10) hold for elements in the subspace since they are in the larger space where the axioms hold. Axiom (5), the existence of the additive inverse follows from that $-\mathbf{u} = (-1)\mathbf{u}$ is in the subspace.

Ex The set $F = \{(x_1, x_2, x_3);\, x_1 + x_2 + x_3 = 0\}$ is a subspace.
Sol It is a subspace of $\mathbf{R}^3$ since if $\mathbf{x}, \mathbf{y} \in S$ then $\mathbf{z} = \mathbf{x} + \mathbf{y}$ satisfy $z_1 + z_2 + z_3 = x_1 + y_1 + x_2 + y_2 + x_3 + y_3 = x_1 + x_2 + x_3 + y_1 + y_2 + y_3 = 0 + 0 = 0$ and $\mathbf{w} = \alpha\mathbf{x}$ satisfy $w_1 + w_2 + w_3 = \alpha x_1 + \alpha x_2 + \alpha x_3 = \alpha(x_1 + x_2 + x_3) = 0$.
Ex The set $G = \{(x_1, x_2, x_3);\, x_1 + x_2 + x_3 = 1\}$ is not a subspace.
Sol If $\mathbf{x}, \mathbf{y} \in G$ then $\mathbf{z} = \mathbf{x} + \mathbf{z} \notin G$ since $z_1 + z_2 + z_3 = x_1 + y_1 + x_2 + y_2 + x_3 + y_3 = x_1 + x_2 + x_3 + y_1 + y_2 + y_3 = 1 + 1 = 2 \neq 1$

Let $\mathbf{v}_1, ..., \mathbf{v}_n$ be vectors in a vector space V. A sum of the form $\alpha_1\mathbf{v}_1 + ... + \alpha_n\mathbf{v}_n$, where $\alpha_1, ..., \alpha_n$ are scalars, is called a **linear combination** of $\mathbf{v}_1, ..., \mathbf{v}_n$. The set of all linear combinations of a $\mathbf{v}_1, ..., \mathbf{v}_n$ is called the **span** of $\mathbf{v}_1, ..., \mathbf{v}_n$ and is denoted by $\mathrm{Span}(\mathbf{v}_1, ..., \mathbf{v}_n)$. The set $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ **span** (is a **spanning set** for) V if every vector in V can be written as a linear combination of $\mathbf{v}_1, ..., \mathbf{v}_n$.

Ex The plane $x_1 + x_2 + x_3 = 0$ is the span of the vectors $(-1, 1, 0)^T$ and $(-1, 0, 1)^T$.
Sol $x_1 + x_2 + x_3 = 0$ is in reduced row echelon form so x_2 and x_3 are free and the solution is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

Th If $\mathbf{v}_1, ..., \mathbf{v}_n$ are in a vector space V, then $\mathrm{Span}(\mathbf{v}_1, ..., \mathbf{v}_n)$ is a subspace of V.
Pf The proof is essentially the same as that the plane $x_1 + x_2 + x_3 = 0$ is a subspace. Let $\mathbf{v} = \alpha_1\mathbf{v}_1 + ... + \alpha_n\mathbf{v}_n$ be an arbitrary element in $\mathrm{Span}\,(\mathbf{v}_1, ..., \mathbf{v}_n)$. then $\beta\mathbf{v} = (\beta\alpha_1)\mathbf{v}_1 + ... + (\beta_n\alpha)\mathbf{v}_n$ is in V, since its a linear combination of $\mathbf{v}_1, ..., \mathbf{v}_n$. Similarly if $\mathbf{w} = \beta_1\mathbf{v}_1 + ... + \beta_n\mathbf{v}_n$ then $\mathbf{v} + \mathbf{w} = (\alpha_1 + \beta_1)\mathbf{v}_1 + ... + (\alpha_n + \beta_n)\mathbf{v}_n \in V$.

Null space and Column space.

The **null space** of an $m \times n$ matrix A is the set of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$; Nul $A = \{\mathbf{x} \in \mathbf{R}^n;\ A\mathbf{x} = \mathbf{0}\}$.

Th The null space of an $m \times n$ matrix A is a subspace of $\mathbf{R}^n$.

Pf We must verify the three properties (a), (b), (c) in the definition of subspace.

(a) $\mathbf{0} \in$ Nul A since $A\mathbf{0} = \mathbf{0}$.

(b) If $\mathbf{u}, \mathbf{v} \in$ Nul A, show that $\mathbf{u} + \mathbf{v} \in$ Nul A. $A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0}$.

(c) If $\mathbf{u} \in$ Nul A, show that $\lambda\mathbf{u} \in$ Nul A. $A(\lambda\mathbf{u}) = \lambda A\mathbf{u} = \lambda\mathbf{0} = \mathbf{0}$.

Ex 1 Find an **explicit description** of Nul A where $A = \begin{bmatrix} 3 & 6 & 6 & 3 & 9 \\ 6 & 12 & 13 & 0 & 3 \end{bmatrix}$.

Sol Row reduction to solve $A\mathbf{x} = 0$; $\begin{bmatrix} 3 & 6 & 6 & 3 & 9 & 0 \\ 6 & 12 & 13 & 0 & 3 & 0 \end{bmatrix} \sim^{(1)/3} \begin{bmatrix} 1 & 2 & 2 & 1 & 3 & 0 \\ 6 & 12 & 13 & 0 & 3 & 0 \end{bmatrix}$

$\sim_{(2)-6(1)} \begin{bmatrix} 1 & 2 & 2 & 1 & 3 & 0 \\ 0 & 0 & 1 & -6 & -15 & 0 \end{bmatrix} \sim^{(1)-2(2)} \begin{bmatrix} 1 & 2 & 0 & 13 & 33 & 0 \\ 0 & 0 & 1 & -6 & -15 & 0 \end{bmatrix}$

Hence $A\mathbf{x} = \mathbf{0} \Leftrightarrow \begin{cases} x_1 + 2x_2 + 13x_4 + 33x_5 = 0 \\ x_3 - 6x_4 - 15x_5 = 0 \end{cases}$. x_2, x_4, x_5 are free so the sol. is

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -2x_2 - 13x_4 - 33x_5 \\ x_2 \\ 6x_4 + 15x_5 \\ x_4 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -13 \\ 0 \\ 6 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -33 \\ 0 \\ 15 \\ 0 \\ 1 \end{bmatrix}$$

Hence Nul $A = \text{Span}\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$, is the span of the three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ above.

The **column space** of an $m{\times}n$ matrix $A = [\,\mathbf{a}_1 \ \cdots \ \mathbf{a}_n\,]$ is the set of all linear combinations of its column vectors; Col $A = \text{Span}\{\mathbf{a}_1, \cdots, \mathbf{a}_n\} = \{\mathbf{y};\ \mathbf{y} = A\mathbf{x}, \text{ for some } \mathbf{x}\}$.

Th The column space of an $m \times n$ matrix A is a subspace of $\mathbf{R}^m$.

Pf In the previous section we showed that the span of any set of vectors is a subspace.

Ex 2 Describe the column space of $A = [\,\mathbf{a}_1 \ \ \mathbf{a}_2 \ \ \mathbf{a}_3 \ \ \mathbf{a}_4\,] = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix}$

Sol Col $A = \text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$.

Quest Can the column space of A be spanned by fewer vectors?

Sol We want to solve $A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3 + x_4\mathbf{a}_4 = 0$, since this will give us some relations between the vectors if there is any. Row reduction on the augmented matrix gives that x_2, x_4 are free variables;

$$\begin{bmatrix} 1 & 2 & 0 & 4 & 0 \\ 2 & 4 & -1 & 3 & 0 \\ 3 & 6 & 2 & 22 & 0 \\ 4 & 8 & 0 & 16 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & -1 & -5 & 0 \\ 0 & 0 & 2 & 10 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x_1 + 2x_2 + 4x_4 = 0 \\ x_3 + 5x_4 = 0 \end{cases} \Rightarrow$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2x_2 - 4x_4 \\ x_2 \\ -5x_4 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ -5 \\ 1 \end{bmatrix}$$

If $x_2 = 1$, $x_4 = 0$ then $-2\mathbf{a}_1 + \mathbf{a}_2 = \mathbf{0}$. If $x_2 = 0$, $x_4 = 1$ then $-4\mathbf{a}_1 - 5\mathbf{a}_3 + \mathbf{a}_4 = \mathbf{0}$.

Quest Can we find a linearly independent subset of $\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$ that span Col A? Using the **linear dependency relations** $-2\mathbf{a}_1 + \mathbf{a}_2 = \mathbf{0}$ and $-4\mathbf{a}_1 - 5\mathbf{a}_3 + \mathbf{a}_4 = \mathbf{0}$ we see that $\mathbf{a}_2 = 2\mathbf{a}_1$ and $\mathbf{a}_4 = 4\mathbf{a}_1 + 5\mathbf{a}_3$. If $\mathbf{y} \in \text{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\}$ then $\mathbf{y} = c_1\mathbf{a}_1 + c_2\mathbf{a}_2 + c_3\mathbf{a}_3 + c_4\mathbf{a}_4 = ?\mathbf{a}_1 + ?\mathbf{a}_3$. Do $\{\mathbf{a}_1, \mathbf{a}_3\}$ span? Are they linearly indep.?