Lecture 5: 2.2 Solving $A\mathrm{x} = 0$ and $A\mathrm{x} = \mathbf{b}$. Row echelon form, pivot variables and free variables, the rank of a matrix.

How do we find all solutions to $A\mathbf{x} = \mathbf{b}$? If A is invertible then the solution is given by $\mathbf{x} = A^{-1}\mathbf{b}$. If A is not invertible there might not be any solution or there might be infinitely many solutions, in which case we want to find them all. If we have a particular solution to $A\mathbf{x}_p = \mathbf{b}$ them we can get the general solution by adding any solution to $A\mathbf{x}_0 = \mathbf{0}$, $A(\mathbf{x}_0 + \mathbf{x}_p) = \mathbf{b}$. We will first study all solutions to $A\mathbf{x} = 0$, when A is not invertible. Then we will study for which $\mathbf{b}$, $A\mathbf{x} = \mathbf{b}$ has a solution.

The set of all solutions are called the nullspace of A and denoted by $N(A)$.

Ex Find the nullspace of $A = \begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix}$.

Sol $\begin{bmatrix} 1 & 2 & 2 & 2 \\ 2 & 4 & 6 & 8 \\ 3 & 6 & 8 & 10 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 2 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 2 & 2 \\ 0 & 0 & 2 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & -2 \\ 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} = U.$

The reduced row echelon form of A has two free variables x_2 and x_4 that can be set to any value and we can solve for the other variables: $\begin{cases} x_1 + 2x_2 - 2x_4 = 0 \\ x_3 + 2x_4 = 0 \end{cases}$.

This gives the general solution $\mathbf{x} = \begin{bmatrix} 2 \\ 0 \\ -2 \\ 1 \end{bmatrix} x_4 + \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} x_2$.

The null space of A is therefore spanned by the two vectors above. Since these are not parallel it is not spanned by only one of them.

A matrix is said to be in **Row Echelon Form** ("step-like form") if
Each **leading entry** (i.e. left most nonzero entry) of a row is in a column to the right of the leading entry of the row above it, e.g.

$$\begin{bmatrix} 1 & * & * & * & * \\ 0 & 1 & * & * & * \\ 0 & 0 & 0 & 1 & * \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

A matrix is said to be in **Reduced Row Echelon Form** if
(i) It is in row echelon form, and (ii) Each leading non-zero entry is 1 and
(iii) The leading entry in each row is the only non-zero entry in its column, e.g.

$$\begin{bmatrix} 1 & 0 & * & 0 & 0 \\ 0 & 1 & * & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Recall the **Elementary Row Operations** on a matrix are
1. Add to one row a multiple of another.
2. Interchange two rows
3. Multiply all entries in a row by the same nonzero number.
Two matrices are said to be **row equivalent** if one can be transformed into the other by elementary row operations.
If the augmented matrices for two systems are row equivalent then they have the same solution set, i.e. elementary row operations don't change the solution set.
Each matrix is row-equivalent to a unique matrix in reduced row echelon form.

A **pivot position** in a matrix is a place corresponding to a leading 1 in the reduced row echelon form. A **pivot column** is a column that contains a pivot position. The **leading variables** or **pivot variables** are the variables corresponding to the pivot columns and the **free variables** are the other variables.

$$\begin{bmatrix} 1 & 0 & * & 0 & 0 \\ 0 & 1 & * & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\begin{matrix} \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \end{matrix}$$
pivot free pivot

The system $A\mathbf{x} = \mathbf{0}$ will have a nontrivial solution $\mathbf{x} \neq \mathbf{0}$ if the reduced row echelon form of A have free variables.

The process of using row operations to transform a matrix to (reduced) row echelon form is generally known as Gaussian elimination, although it turned out the Chinese were using this method 2000 years earlier.

Gauss is one of the greatest mathematicians of all times with fundamental contributions to number theory, astronomy and geometry. He lived in Germany 1777-1855. He is said to have been able to do arithmetic before he could speak. At 3 he corrected a mistake in the payroll for his fathers company but his father didn't think much of his genius.

How Gauss developed his elimination method is noteworthy. An astronomer Piazzi discovered what he believed was a new planet and was able to observe its path for only 40 days. From these limited observations Gauss was able to predict were the astroid would return a year later. In the course of his computations Gauss had to solve a system of 17 linear equations. In dealing with this problem he also used the method of least square approximation that he previously developed. We will learn this method later in the course. Since Gauss at first refused to reveal his method some people accused him of sorcery.

Ex For which $\mathbf{b}$ can $A\mathbf{x} = \mathbf{b}$ be solved, i.e. which $\mathbf{b}$ are in the column space of A?
Sol We perform row reductions on the augmented matrix

$$\left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 2 & 4 & 6 & 8 & b_2 \\ 3 & 6 & 8 & 10 & b_3 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 2 & 4 & b_3 - 3b_1 \end{array}\right] \rightarrow \left[\begin{array}{cccc|c} 1 & 2 & 2 & 2 & b_1 \\ 0 & 0 & 2 & 4 & b_2 - 2b_1 \\ 0 & 0 & 0 & 0 & b_3 - b_1 - b_2 \end{array}\right]$$

In order for the system $A\mathbf{x} = \mathbf{b}$ to be solvable we must have that $b_3 - b_1 - b_2 = 0$. The column space is spanned by the 4 colkumn vectors of A, but can it be spanned by fewer of these vectors. First since its a set in $\mathbf{R}^3$ it can be spanned by 3 vectors but since its a plane it can actually be spanned by two vectors. Which ones will work? We claim that the columns corresponding to the pivot variables will always work. First you don't need any of the other column vectors since the free columns in the reduced row echelon form can be written as linear combinations of the pivot columns and since the same relations between the columns hold in the original form. Secondly, the pivot columns can not be written as linear combinations of each other in the reduced row echelon form and hence not in the original form.

Suppose that A is an $m \times n$ matrix. We define the **rank** of A to be the number of pivots in the reduced row echelon form and denote it by $r = \text{rank}(A)$. The rank is the number of equations that there really was in the system after we reduced away the equations that were consequences of the others.

We note that the system $A\mathbf{x} = \mathbf{b}$ is always solvable if it has full rank $r = m$.
If $r < m$ there will be some $\mathbf{b}$ for which there is no solution just as in the example.
We also note that if $r < n$ the homogeneous system $A\mathbf{x} = 0$ will have nontrivial solutions $\mathbf{x} \neq \mathbf{0}$ since there will be at least one free variable.