Lecture 6: Linear Independence, Spanning, Basis and Dimension.
The homogeneous system $A\mathbf{x} = \mathbf{0}$ can be studied from a different perspective by writing them as vector equations;

$$(1)\quad \begin{bmatrix} 1 & 2 & -3 \\ 3 & 5 & 9 \\ 5 & 9 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad x_1 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

A set of vectors $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ in $\mathbf{R}^n$ is said to be **linearly independent** if

$$x_1\mathbf{v}_1 + \cdots + x_p\mathbf{v}_p = \mathbf{0}$$

has only the trivial solution $x_1 = \cdots = x_p = 0$. The set $\{\mathbf{v}_1, \dots, \mathbf{v}_p\}$ is said to be **linearly dependent** if there are weights $\lambda_1, \dots, \lambda_p$ not all 0, such that

$$\lambda_1\mathbf{v}_1 + \cdots + \lambda_p\mathbf{v}_p = \mathbf{0}.$$

Ex 1 Determine if $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix}$ are linearly independent?
Sol They are linearly dependent if (1) has a nontrivial solution. Augmented:

$$\begin{bmatrix} 1 & 2 & -3 & 0 \\ 3 & 5 & 9 & 0 \\ 5 & 9 & 3 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & -1 & 18 & 0 \\ 0 & -1 & 18 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & -3 & 0 \\ 0 & -1 & 18 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 33 & 0 \\ 0 & -1 & 18 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since x_3 is free there are nontrivial solutions $x_1 = -33x_3$, $x_2 = 18x_3$, x_3 is free. If we e.g. let $x_3 = 1$ then $x_1 = -33$ and $x_2 = 18$ so we have the linear dependence relation

$$-33 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + 18 \begin{bmatrix} 2 \\ 5 \\ 9 \end{bmatrix} + 1 \begin{bmatrix} -3 \\ 9 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \Leftrightarrow \quad \begin{bmatrix} 1 & 2 & -3 \\ 3 & 5 & 9 \\ 5 & 9 & 3 \end{bmatrix} \begin{bmatrix} -33 \\ 18 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The columns of A are linearly independent $\Leftrightarrow$ $A\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Th n vectors $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ in $\mathbf{R}^m$ are linearly dependent if $n > m$.
Pf $A = [\,\mathbf{v}_1 \cdots \mathbf{v}_n\,]$ is a $m \times n$ matrix. If $n > m$ then the homogeneous system $A\mathbf{x} = \mathbf{0}$ has more unkonowns than equations so there must be free variables. Hence $A\mathbf{x} = \mathbf{0}$ has a nontrivial solution $\mathbf{x} \neq \mathbf{0}$ so the columns of A are linearly dependent.

Ex 2 With as little work as possible decide if the following sets of vectors are linearly dependent. (a) $\left\{ \begin{bmatrix} 3 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 9 \\ 6 \\ 4 \end{bmatrix} \right\}$ and (b) the columns of $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & 8 & 9 & 0 \\ 9 & 8 & 7 & 6 & 5 \end{bmatrix}$.
Sol (a) linearly independent since one is not a multiple of the other. (b) linearly dependent since there are more columns than entries in the vectors.

We say that a set of vectors $\{\mathbf{v}_1, \cdots, \mathbf{v}_n\}$ **span** H if every $\mathbf{v} \in H$ can be written as a linear combination of them; $\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_n\mathbf{v}_n$.

Ex 3 Do the vectors in Ex. 2 or Ex. 3 Span $\mathbf{R}^3$?

Basis.
Let H be a subspace of a vector space V. $\{\mathbf{b}_1, \cdots, \mathbf{b}_n\}$ is called a **basis** for H if
(i) $\{\mathbf{b}_1, \cdots, \mathbf{b}_n\}$ are linearly independent, and (ii) $\{\mathbf{b}_1, \cdots, \mathbf{b}_n\}$ span H.
This means that every $\mathbf{v} \in H$ can be written $\mathbf{v} = c_1\mathbf{b}_1 + \cdots + c_n\mathbf{b}_n$ for a unique choose of constant $c_1, \ldots, c_n$.

Ex 4 Show that $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is a basis for $\mathbf{R}^3$, the **standard basis**

Sol They span $\mathbf{R}^3$ since any vector can be written $\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3$.
They are linearly independent since $x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + x_3\mathbf{e}_3 = \mathbf{x} = \mathbf{0}$ implies $x_1 = x_2 = x_3 = 0$.

Ex 5 Show that $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}$ is a basis for $\mathbf{R}^3$.

Sol $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ are linearly independent and span $\mathbf{R}^3$ if and only if the reduced row echelon form of the matrix $A = [\,\mathbf{v}_1\ \mathbf{v}_2\ \mathbf{v}_3\,]$ has 3 pivot columns. Row reduction

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 0 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 1 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -2 \\ 0 & 0 & 5 \end{bmatrix}$$

Question: Why lin. indep. if 3 pivots? ($A\mathbf{x} = \mathbf{0}$) Why span if 3 pivots? ($A\mathbf{x} = \mathbf{b}$)

Ex 6 Explain why not basis: (a) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 7 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -3 \\ 7 \end{bmatrix} \right\}$, (b) $\left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} \right\}$.

Sol (a) Linearly dependent since more vectors than rows. (b) Since the matrix with the two vectors as columns does not have a pivot in every row they do not span $\mathbf{R}^3$.

The Spanning Theorem A basis can be constructed from a spanning set by discarding vectors which are linear combinations of the preceding vectors in the set.

Ex 7 Find a basis for $\mathrm{Col}\,A$, where $A = [\mathbf{a}_1 \mathbf{a}_2\, \mathbf{a}_3\, \mathbf{a}_4]$ is given below:

$$[\mathbf{a}_1 \mathbf{a}_2\, \mathbf{a}_3\, \mathbf{a}_4] = \begin{bmatrix} 1 & 2 & 0 & 4 \\ 2 & 4 & -1 & 3 \\ 3 & 6 & 2 & 22 \\ 4 & 8 & 0 & 16 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 4 \\ 0 & 0 & 1 & 5 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = [\mathbf{b}_1 \mathbf{b}_2\, \mathbf{b}_3\, \mathbf{b}_4]$$

$\mathbf{b}_2 = 2\mathbf{b}_1$ and $\mathbf{a}_2 = 2\mathbf{a}_1$
$\mathbf{b}_4 = 4\mathbf{b}_1 + 5\mathbf{b}_3$, $\mathbf{a}_4 = 4\mathbf{a}_1 + 5\mathbf{a}_3$
$\mathbf{b}_1$ and $\mathbf{b}_3$ are not multiples.
$\mathbf{a}_1$ and $\mathbf{a}_3$ are not multiples.

Elementary row operations do not affect linear dependency relations among the columns, since $A\mathbf{x} = \mathbf{0} \Leftrightarrow B\mathbf{x} = \mathbf{0}$. Hence $\mathrm{Span}\{\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4\} = \mathrm{Span}\{\mathbf{a}_1, \mathbf{a}_3\}$.

Th The pivot columns of A form a basis for Col A.

To form a basis for Nul A, use row operations to find the reduced row echelon form $[\,A\ \mathbf{0}\,] \sim [\,R\ \mathbf{0}\,]$ and use it to find the general solution of $A\mathbf{x} = \mathbf{0}$ in parametric form. The vectors found in parametric form is a basis.
To find a basis for $\mathrm{Col}\,A$, use row operations to find the reduced row echelon form $A \sim R$ and use it to find the pivot columns. The pivot columns of A form a basis.

Dimension.

Recall that $\mathbf{b}_1, ..., \mathbf{b}_n$ form a **basis** for a vector space V if

(i) $\mathbf{b}_1, ..., \mathbf{b}_n$ are linearly independent, and (ii) $\mathbf{b}_1, ..., \mathbf{b}_n$ span V.

Th If $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ is a spanning set for V then any collection of p vectors $\{\mathbf{u}_1, ..., \mathbf{u}_p\}$, where $p > n$, is linearly dependent.

Pf Since $\mathbf{v}_1, ..., \mathbf{v}_n$ span V we can write each $\mathbf{u}_i$ as a linear combination:

$$\mathbf{u}_i = a_{i1}\mathbf{v}_1 + ... + a_{in}\mathbf{v}_n$$

A linear combination of the $\mathbf{u}_i$ can be written

$$\begin{aligned} c_1\mathbf{u}_1 + ... + c_p\mathbf{u}_p &= c_1\big(a_{11}\mathbf{v}_1 + \cdots + a_{1n}\mathbf{v}_n\big) + \cdots + c_p\big(a_{p1}\mathbf{v}_1 + \cdots + a_{pn}\mathbf{v}_n\big) \\ &= \big(c_1a_{11} + \cdots + c_pa_{p1}\big)\mathbf{v}_1 + \cdots + \big(c_1a_{p1} + \cdots + c_pa_{pn}\big)\mathbf{v}_n \end{aligned}$$

Hence $c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p = \mathbf{0}$ if

$$\begin{aligned} c_1a_{11} + \cdots + c_pa_{p1} &= 0 \\ &\vdots \\ c_1a_{1n} + \cdots + c_pa_{pn} &= 0 \end{aligned}$$

Since $p > n$ this is a homogenous system for $c_1, \cdots, c_p$ with more unknowns than equations so it has a nontrivial solution. Hence there are constants $c_1, ..., c_p$ not all zero such that $c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p = \mathbf{0}$, i.e. $\mathbf{u}_1, ..., \mathbf{u}_p$ are linearly dependent.

Cor If $\{\mathbf{v}_1, ..., \mathbf{v}_n\}$ and $\{\mathbf{u}_1, ..., \mathbf{u}_m\}$ are bases for V then $n = m$.

The number of elements in a basis for V is called the **dimension** of V, written $\dim V$.

Dimensions of the column and null spaces of a matrix.

Ex 8 Find $\dim \mathrm{Col}\, A$ and $\dim \mathrm{Nul}\, A$, where $A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 8 \end{bmatrix}$.

Sol $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 7 & 8 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ so the pivot columns $\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 3 \\ 7 \end{bmatrix} \right\}$ is a basis for Col A and $\dim \mathrm{Col}$ A = 2.

To find Nul A we solve $A\mathbf{x} = \mathbf{0}$:

$\begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\ 2 & 4 & 7 & 8 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{bmatrix} \Rightarrow \begin{cases} x_1 = -2x_2 - 4x_4 \\ x_3 = 0 \end{cases}$ and

$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ so $\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for Nul A and $\dim \mathrm{Nul}\, A = 2$.

$\dim \mathrm{Col}\, A$ = number of pivot columns of A, $\dim \mathrm{Nul}\, A$ = number of free variables of A