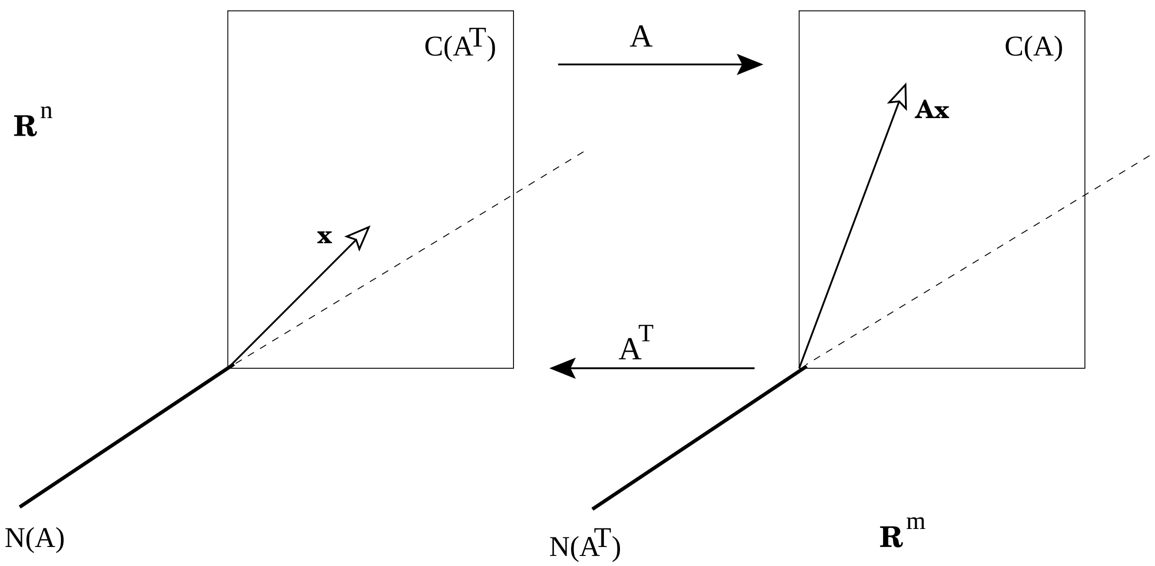
Lecture 7: 2.4 The Four Fundamental Subspaces.

There are four fundamental subspaces associated with an $m \times n$ matrix A:

1. The column space $C(A)$.
2. The Null space $N(A)$.
3. The Row space which is the same as the column space of the transpose $C(A^T)$.
4. The left null space which is the same as the null space of the transpose $N(A^T)$.

The matrix can be considered as a map $A : \mathbf{R}^n \to \mathbf{R}^m$, with the image given by matrix multiplication: If $A = [\mathbf{a}_1, \dots, \mathbf{a}_n]$, where $\mathbf{a}_i \in \mathbf{R}^m$ are the columns of A, and $\mathbf{x} \in \mathbf{R}^n$, then $A\mathbf{x} = \mathbf{a}_1 x_1 + \cdots + \mathbf{a}_n x_n \in \mathbf{R}^m$. Similarly $A^T : \mathbf{R}^m \to \mathbf{R}^n$.
$N(A)$ and $C(A^T)$ are subsets of $\mathbf{R}^n$ whereas $C(A)$ and $N(A^T)$ are subsets of $\mathbf{R}^m$.



We want to find bases for all the four spaces. In order to do so we will have to make row reduction. Let us therefore look on an example

$$A = \begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 1 & 2 & 1 \\ 3 & 2 & 3 & 1 \end{bmatrix} \to \cdots \to \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

$$\uparrow \quad \uparrow \quad \uparrow \quad \uparrow$$

pivot free

where R is the reduced row echelon form of A. Recall that the pivot elements are the elements with a 1 in leftmost nonzero position in a row and the pivot columns are the columns with a pivot element in them. The other columns are the free columns. The rank r of a matrix is defined to be the number of pivots in the reduced row echelon form R, In the case above $r = 2$.

3. We claim that the row space of A is equal to the row space of R. First the row space of R is clearly contained in the row space of A, because the rows in R are formed by linear combinations of the rows of A. However, these row operations are reversible so the rows of A are linear combinations of the rows in R and so the row space of A is also contained in the row space of R. A basis for the row space is therefore just the nonzero rows of R and the dimension of the row space is r.

Instead of talking about the row space we prefer to talk about the columns space of A^T. In our case $\begin{bmatrix}1\\0\\1\\1\end{bmatrix}$ and $\begin{bmatrix}0\\1\\1\\0\end{bmatrix}$ form a basis for $C(A^T)$ and the dimension is $r = 2$.

2. The nullspace of A is equal to the nullspace of R. Recall that we can write the solutions to $R\mathbf{x} = 0$ in terms of the free variables. Expressed differently, if we set one of the free variables to 1 and the others to 0 we get a solution. For each of the free variables we therefore get a a solution and these vectors are linearly independent since each is 1 at a place where the others are 0 so they form a basis. Since there are $n - r$ free variables the dimension of the nullspace is $n - r$. In the example if we first set $x_4 = 1$ and $x_3 = 0$, and then $x_4 = 0$ and $x_3 = 1$, we get the solutions $\begin{bmatrix}-1\\0\\0\\1\end{bmatrix}$ and $\begin{bmatrix}-1\\-1\\1\\0\end{bmatrix}$, and the dimension of the nullspace is $n - r = 4 - 2 = 2$.

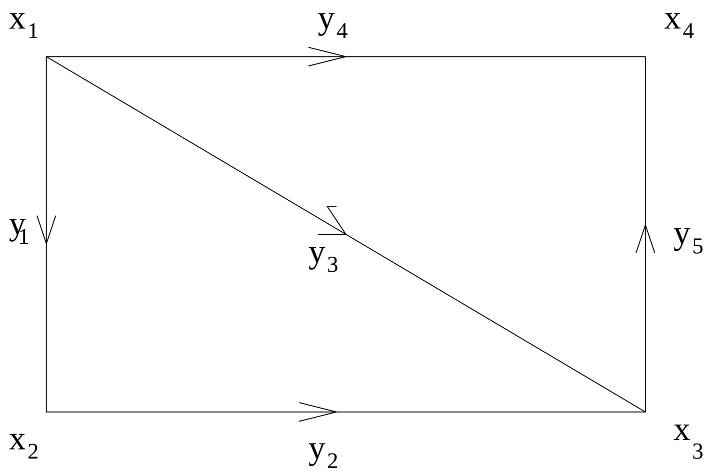
Note that the nullspace and the row space are orthogonal to each other, i.e. each of the basis vectors for the nullspace are orthogonal to each of the basis vectors for the row space and so this is true for each pairs of vectors from the two spaces. This is just an expression of the fact that each equation in the system $A\mathbf{x} = 0$ is an inner product of a row of A with the vector $\mathbf{x}$.

1. The column space $C(A)$. Note that the column space of A is not the same as the column space of R so a basis for $C(R)$ is not a basis for $C(A)$. Still its a beautiful fact that since the pivot columns of R form a basis for $C(R)$ the corresponding columns of A form a basis for $C(A)$. To prove this fact we first note that the free columns can be expressed in terms of the pivot columns. In or case if $R = [\mathbf{r}_1\ \mathbf{r}_2\ \mathbf{r}_3\ \mathbf{r}_4]$, where $\mathbf{r}_i$ are the columns of R, $A = [\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3\ \mathbf{a}_4]$ where $\mathbf{a}_i$ are the columns of A, then $\mathbf{r}_1 - \mathbf{r}_4 = 0$ and $\mathbf{r}_1 + \mathbf{r}_2 - \mathbf{r}_3 = 0$, and we claim that also $\mathbf{a}_1 - \mathbf{a}_4 = 0$ and $\mathbf{a}_1 + \mathbf{a}_2 - \mathbf{a}_3 = 0$. Why is this so? The answer is obvious that you can't see it. A relation of the form $x_1\mathbf{r}_1 + \cdot + x_n\mathbf{r}_n = \mathbf{0}$ is nothing but $R\mathbf{x} = 0$, which we know is equivalent to $A\mathbf{x} = 0$, which in turn is equivalent to $x_1\mathbf{a}_1 + \cdot + x_n\mathbf{a}_n = \mathbf{0}$. This shows that every vector ion the column space of A can be expressed in terms of $\mathbf{a}_1, \mathbf{a}_2$. That this are independent follow from the fact that $\mathbf{r}_1, \mathbf{r}_2$ are, since if there was a relation $x_1\mathbf{a}_1 + x_2\mathbf{a}_2 = \mathbf{0}$, there would also be a relation $x_1\mathbf{r}_1 + x_2\mathbf{r}_2 = \mathbf{0}$, which is not possible since $\mathbf{r}_1, \mathbf{r}_2$ are independent. In our case $\begin{bmatrix}1\\1\\1\end{bmatrix}$ and $\begin{bmatrix}2\\1\\2\end{bmatrix}$ form a basis for $C(A)$ and the dimension is $r = 2$.

4. The left nullspace $N(A^T)$. This is some work but in principle we should be able to do row reduction on A^T to find its reduced row echelon form R'.

$$A = \begin{bmatrix}1 & 1 & 1\\2 & 1 & 2\\3 & 2 & 3\\1 & 1 & 1\end{bmatrix} \to \cdots \to \begin{bmatrix}1 & 0 & 1\\0 & 1 & 0\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix} = R'$$

2.5 Graphs and Networks.



The graph has four nodes x_1, x_2, x_3, x_4 and five edges y_1, y_2, y_3, y_4, y_5. The edges are directed. The incident matrix is 5×4 matrix which has 1 in the ith column and jth row if the edge y_j goes into the node x_i, -1 if it goes out from the node and 0 if it dos not connect to the node.

$$A = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ -1 & 0 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & -1 & 1 \end{bmatrix}$$

The nullspace of A; $A\mathbf{x} = \mathbf{0}$ consist of vectors $\mathbf{x} = c \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. This has a meaning if we think of x_1, x_2, x_3, x_4 as the potentials at the nodes. The components of A gives the potential differences but we can not measure the absolute potential.

The column space of A. For which differences can we solve $A\mathbf{x} = \mathbf{b}$? What has to hold is that the sum of potentials around any loop has to add up to 0. This is called Kirchoff's Voltage law.

Left Nullspace $A^T\mathbf{y} = \mathbf{0}$. This corresponds to that the total current into every node is zero.b This is called Kirchoffs's Current law.

The row space of A.