Lecture 8: 2.6 Linear Transformations.

A **transformation** $T: W \to V$ from one vector space W to another V, is called **linear** if $T\big(\mathbf{x}+\mathbf{y}\big) = T(\mathbf{x}) + T(\mathbf{y})$ and $T(\lambda \mathbf{x}) = \lambda T(\mathbf{x})$, which can alternatively be summarized in $T(\lambda_1 \mathbf{x}_1 + \lambda_2 \mathbf{x}_2) = \lambda_1 T(\mathbf{x}_1) + \lambda_2 T(\mathbf{x}_2)$.

Matrix multiplication by an $m \times n$ matrix A gives a mapping $\mathbf{R}^n \ni \mathbf{x} \to A\mathbf{x} \in \mathbf{R}^m$:

$$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} \to \begin{bmatrix} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n \end{bmatrix}.$$

A **matrix transformation** $T(\mathbf{x}) = A\mathbf{x}$ is the simplest type of linear transformation.

Ex 1 Let $T: \mathbf{R}^2 \to \mathbf{R}^2$ be the projection onto the x_1 axis; $T: \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} x_1 \\ 0 \end{bmatrix}$. Then its easy to check that T is a linear transformation. Moreover, T can in fact be given by matrix multiplication. This is seen by expressing T in terms of a basis:

$$T: \begin{bmatrix} 1 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_2 \to \begin{bmatrix} 1 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 0 \end{bmatrix} x_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

In general if $\mathbf{x} \in \mathbf{R}^n$ we can write

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + x_n \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = x_1 \mathbf{e}_1 + \cdots + x_n \mathbf{e}_n$$

and by repeated use of linearity

$$T(\mathbf{x}) = T\big(x_1 \mathbf{e}_1 + \cdots + x_n \mathbf{e}_n\big) = x_1 T(\mathbf{e}_1) + \cdots + x_n T(\mathbf{e}_n)$$

This is the matrix multiplication of the matrix $[\,T(\mathbf{e}_1) \quad \cdots \quad T(\mathbf{e}_n)\,]$ and $\mathbf{x}$ so

If T is a linear transformation $\mathbf{R}^n \to \mathbf{R}^m$ then $T(\mathbf{x}) = A\mathbf{x}$, where $A = \big[\mathbf{a}_1 \cdots \mathbf{a}_n\big]$ is the matrix whose j:th column is $\mathbf{a}_j = T(\mathbf{e}_j)$ and $\mathbf{e}_j$ is the vector with 1 in the jth place and 0 otherwise.

Ex 2 Let $T: P_3 \to P_3$ be the transformation acting on P_3 polynomials of degree ≤ 3 by differentiation $Tp(t) = p'(t)$. Is T a linear transformation? Can it be given as matrix multiplication?

Sol Yes it is a linear transformation, because $\big(\alpha p(t) + \beta q(t)\big)' = \alpha p'(t) + \beta q'(t)$. It can be given as matrix multiplication if we introduce a basis. A basis for the polynomials are the monomials $1, t, t^2, t^3 \ldots$, We can write $p(t) = c_1 \cdot 1 + c_2 t + c_3 t^2 + c_4 t^3$. Then $p'(t) = c_2 + 2c_3 t + 3c_4 3t^2$. The polynomial $p(t)$ corresponds to coordinate vector $\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$ and the polynomial $p'(t) = d_1 \cdot 1 + d_2 t + d_3 t^2 + d_4 t^3$ corresponds to the vector $\begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} = \begin{bmatrix} c_2 \\ 2c_3 \\ 3c_4 \\ 0 \end{bmatrix}$. so we have $\begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$

Ex 3 Rotating an angle θ counterclockwise in the plane is a linear transformation.
Pf That $T(\alpha \mathbf{x}) = \alpha T(\mathbf{x})$ says that it is the same thing to first multiply by α and then rotate as it is to first rotate and then multiply by α, which is clear from the geometric definition of scalar multiplication. That $T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y})$ says that it the same thing to first add the vectors and then rotate the sum as it is to first rotate them and then add the results. This follows from the geometric definition of addition with the parallelogram law, since rotating the whole parallelogram by an angle θ rotates all its sides by the same angle.

Ex 4 Rotating the angle θ is given by multiplying by the matrix $Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

Pf Since it is a linear it follows that $Q = [\mathbf{q}_1\ \mathbf{q}_2]$, where $\mathbf{q}_1 = T(\mathbf{e}_1)$ and $\mathbf{q}_2 = T(\mathbf{e}_2)$. The rotation of $\mathbf{e}_1$ by an angle θ is the vector $\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}$ and rotation of $\mathbf{e}_2$ is $\begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}$.

If B is an $n \times k$ matrix then multiplication first by B and then A

$$\mathbf{x} \xrightarrow{\text{multiply by } B} B\mathbf{x} \xrightarrow{\text{multiply by } A} A(B\mathbf{x})$$

defines a map $\mathbf{R}^k \ni \mathbf{x} \to A(B\mathbf{x}) \in \mathbf{R}^m$. The matrix product AB is constructed so that multiplying by the matrix AB

$$\mathbf{x} \xrightarrow{\qquad \text{multiply by } AB \qquad} (AB)\mathbf{x}$$

is the same as first multiplying by B and then by A, i.e. $(AB)\mathbf{x} = A(B\mathbf{x})$.

Let us conclude the discussion by an example using that composition of linear maps corresponds to matrix multiplication:
Ex 5 Find the matrix for the linear transformation obtained by first scaling a vector by a factor 3 and then rotating it by an angle $-\pi/2$ counter clockwise.
Sol $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -x_2 \\ x_1 \end{bmatrix}$ **rotates** vectors an angle $\pi/2$ counterclockwise.
$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3x_1 \\ 3x_2 \end{bmatrix}$ **scales** vectors by a factor 3.
$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & -3 \\ 3 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -3x_2 \\ 3x_1 \end{bmatrix}$ scales and rotates.