MATH 132A PARTIAL DIFFERENTIAL EQUATIONS

Lecture 1: Introduction.

Notation Let $u\,(t, \mathbf{x})$ denote a function of time t and n space variables $\mathbf{x} = (x_1, \dots, x_n)$ or x, y, z. Let $\partial_{x_i} u = \dfrac{\partial u}{\partial x_i}$, or u_x, u_y, u_z be the partial derivative of u with respect to x_i.

Examples. Partial Differential equations arise in e.g. physics and geometry:
Simplest first order equation Transport equation

$$\partial_t u + \sum_{i=1}^{n} b^i \partial_{x_i} u = 0$$

Classical Second order constant coefficient Linear equations: Laplace equation:

$$\triangle u \equiv \sum_{i=1}^{n} \partial_{x_i}^2 u = 0$$

Wave equation (e.g. the equation of a string)

$$\Box u \equiv \partial_t^2 u - \triangle u = 0$$

Heat equation (temperature distribution and as well as diffusion of gas)

$$\partial_t u - \triangle u = 0$$

Schroedinger equation (quantum mechanics electrons in an atom)

$$i \partial_t u + \triangle u = 0$$

Nonlinear equations Burgers' equation (shock waves in fluids)

$$\partial_t u + u \partial_x u = 0$$

Linear Systems Maxwell's equations (electromagnetism)

$$\begin{cases} \mathbf{E}_t = \textbf{curl}\,\mathbf{B} \\ \mathbf{B}_t = -\textbf{curl}\,\mathbf{E} \\ \operatorname{div}\mathbf{B} = \operatorname{div}\mathbf{E} = 0 \end{cases}$$

Systems of ordinary differential equations

$$\frac{d}{dt}\mathbf{u} + A\mathbf{u} = 0, \qquad \mathbf{u} = (u_1, \dots, u_N) \text{ vector}, \qquad A = (a_{ij}) \quad N \times N \text{ matrix}$$

Nonlinear systems Euler's equations of an incompressible fluid

$$\begin{cases} \partial_t u_i + \sum_{k=1}^{n} u_k \partial_{x^k} u_i = -\partial_i p \\ \sum_{i=1}^{n} \partial_{x^i} u_i = 0 \end{cases}$$

Einstein's general relativistic equations for the gravitational field represented by the metric tensor $\{g_{\alpha\beta}\}_{\alpha,\beta=0,1,2,3}$, of space time is that the Ricci curvature vanishes:

$$R_{\mu\nu}(g) = 0$$

which in harmonic coordinates becomes a system of nonlinear wave equations

$$\Box_g g_{\mu\nu} = F_{\mu\nu}(g, \partial g), \qquad \Box_g = \sum_{\alpha,\beta=0,1,2,3} g^{\alpha\beta} \partial_{x^\alpha} \partial_{x^\beta}$$

Evolution equations. The wave, heat, Schroedinger, transport and ordinary differential equations are *evolution* equations describing evolving phenomena. For evolution equations we want to find a solution for future times from knowledge of initial conditions.

Stationary equations. Laplace equation is a *stationary equation.* For stationary equations we want to find a solution in the interior of a domain from boundary conditions.

Strategies for Solving PDE's. Linear constant coefficient PDEs can be solved more or less explicitly. For nonlinear equations we can in general not find an explicit solution but instead we just ask if the problem is *well posed*, i.e. if:

(a) the problem has a solution,
(b) the solution is unique,
(c) the solution depends continuously on data in a certain class.

For nonlinear equations one can usually prove local existence of a solution but the solution might blow up after some time, e.g. black holes in general relativity or shocks in fluids.

The evolution equations and Fourier series. Let us consider the simplest case of solving the linear wave equations on a circle:

$$\partial_t^2 u - \partial_x^2 u = 0, \qquad u(0,x) = f(x), \quad u_t(0,x) = g(x), \tag{1.1}$$

were data are assumed to be periodic $f(x+2\pi) = f(x)$ and $g(x+2\pi) = g(x)$. We are looking for solution $u(t,x)$ that is periodic in space $u(t,x+2\pi) = u(t,x)$. (This is a simplified version of looking for solutions to the boundary problem with boundary conditions $u(t,0) = u(t,2\pi) = 0$, which is the equation of a string.) Periodic functions can be expanded in a Fourier series

$$u(t,x) = \sum_{k=-\infty}^{\infty} c_k(t)\, e^{ikx}. \tag{1.2}$$

If you don't know this fact we can just say that we are looking for solutions of this form. If this is to satisfy the wave equation then we must have

$$\partial_t^2 u - \partial_x^2 u = \sum_{k=-\infty}^{\infty} \big(\ddot{c}_k(t) + k^2 c_k(t)\big) e^{ikx} = 0$$

(here $\dot{c}_k(t) = dc_k(t)/dt$.) from which it follows that we must have

$$\ddot{c}_k(t) + k^2 c_k(t) = 0, \tag{1.3}$$

for all k. Solving this linear ordinary differential equation gives

$$c_k(t) = A_k e^{ikt} + B_k e^{-ikt} \tag{1.4}$$

The constants A_k and B_k are determined by expanding initial data in Fourier series

$$u(0,x) = f(x) = \sum_{k=-\infty}^{\infty} d_k e^{ikx}, \qquad u_t(0,x) = g(x) = \sum_{k=-\infty}^{\infty} e_k e^{ikx} \tag{1.5}$$

where

$$c_k(0) = A_k + B_k = d_k, \qquad \dot{c}_k(0) = ik(A_k - B_k) = e_k$$

This shows that (1.2) is the solution to (1.1) for all initial data of the form (1.5). The initial value problem is well posed since there is a unique solution and the coefficients (1.4) at positive times are bounded by the coefficients at the initial time. More precisely the energy is preserved:

$$|\dot{c}_k(t)|^2 + k^2 |c_k(t)|^2 = |\dot{c}_k(0)|^2 + k^2 |c_k(0)|^2$$

(In fact $d\big(|\dot{c}_k|^2 + k^2 |c_k|^2\big)/dt = d\big(\dot{c}_k \bar{\dot{c}}_k + k^2 c_k \bar{c}_k\big)/dt = \dot{c}_k\big(\overline{\ddot{c}_k + k^2 c_k}\big) + \bar{\dot{c}}_k(\ddot{c}_k + k^2 c_k) = 0.$)

Remark: Infinite dimensional dynamical system. One can think of an evolution equation, e.g. the wave equation, as an infinity dimensional system

$$u_{tt} = Au$$

where A is the operator $A = \triangle$ expressed in the infinite dimensional Fourier bases $\{e^{ikx}\}$ as multiplying the coefficient c_k of e^{ikx} by $-k^2$.

Stationary equations. The interesting thing is if we try to solve Laplace equation

$$u_{tt} + u_{xx} = 0$$

with Fourier series, we get

$$c_k(t) = e^{kt} A_k + e^{-kt} B_k$$

so the modes are exponentially growing if say initially $A_k \sim e^{-\sqrt{|k|}}$, which are the Fourier coefficients of a smooth function. The conclusion is that the IVP for Laplace equation is not well-posed. Instead one poses boundary data say when $t = 0$ and $t = 1$. If $c_k(0) = A_k + B_k$ and $c_k(1) = e^k A_k + e^{-k} B_k$ are bounded it follows that $c_k(t)$ will be bounded for, $0 \leq t \leq 1$:

$$|c_k(t)| \leq \max\left(|c_k(0)|, |c_k(1)|\right), \qquad \text{for} \quad 0 \leq t \leq 1.$$