**Lecture 10: Diffusion equation.** We will now derive the fundamental solution for the diffusion equation in three space dimensions:

$$u_t = k\triangle u = k\big(u_{xx} + u_{yy} + u_{zz}\big), \qquad u(\mathbf{x},0) = f(\mathbf{x}). \tag{10.1}$$

We claim that

$$u(\mathbf{x},t) = \frac{1}{(4\pi kt)^{3/2}} \iiint \exp\Big(-\frac{|\mathbf{x}-\mathbf{y}|^2}{4kt}\Big) f(\mathbf{y})\, d\mathbf{y} = \iiint S_3(\mathbf{x}-\mathbf{y},t) f(\mathbf{y})\, d\mathbf{y}, \tag{10.2}$$

solves (10.1). To prove this let

$$S(x,t) = \frac{1}{(4\pi kt)^{1/2}} e^{-x^2/4kt}$$

be the one dimensional source function. Note that we can write the 3 dimensional source function

$$S_3(x,y,z,t) = S(x,t)S(y,t)S(z,t)$$
$$= \frac{1}{(4\pi kt)^{1/2}} e^{-x^2/4kt} \frac{1}{(4\pi kt)^{1/2}} e^{-x^2/4kt} \frac{1}{(4\pi kt)^{1/2}} e^{-x^2/4kt} = \frac{1}{(4\pi kt)^{3/2}} e^{-(x^2+y^2+z^2)/4kt}$$

If we take derivative we get

$$\partial_t\big(S(x,t)S(y,t)S(z,t)\big) = S_t(x,t)S(y,t)S(z,t) + S(x,t)S_t(y,t)S(z,t) + S(x,t)S(y,t)S_t(z,t)$$

and

$$(\partial_x^2+\partial_y^2+\partial_z^2)\big(S(x,t)S(y,t)S(z,t)\big) = S_{xx}(x,t)S(y,t)S(z,t)+S(x,t)S_{yy}(y,t)S(z,t)+S(x,t)S(y,t)S_{zz}(z,t)$$

Since $S$ is the one dimensional source function $S_t(x,t) = S_{xx}(x,t)$, $S_t(y,t) = S_{yy}(y,t)$, $S_t(z,t) = S_{zz}(z,t)$ so we have shown that

$$\partial_t S_3 = k\triangle S_3.$$

It follows that $u$ given by (10.2) is a solution of the diffusion equation in (10.1), but we must show that it satisfies the initial conditions in (10.1) as well. For this we must take a limit as in section 3.5. We need to show that

$$u(\mathbf{x},t) = \frac{1}{(4\pi kt)^{3/2}} \iiint \exp\Big(-\frac{|\mathbf{z}|^2}{4kt}\Big) f(\mathbf{x}+\mathbf{z})\, d\mathbf{z} \to f(\mathbf{x}), \qquad t\to 0, \tag{10.3}$$

where we have made the change of variables $\mathbf{z} = \mathbf{y}-\mathbf{x}$. If we make the further change of variables $\mathbf{y} = \mathbf{z}/\sqrt{4kt}$ we get

$$u(\mathbf{x},t) = \frac{1}{(\pi)^{3/2}} \iiint \exp\big(-|\mathbf{y}|^2\big) f(\mathbf{x}+\mathbf{y}\sqrt{4kt})\, d\mathbf{y}$$

If we subtract off $f(\mathbf{x})$ we get

$$u(\mathbf{x},t) - f(\mathbf{x}) = \frac{1}{\pi^{3/2}} \iiint \exp\big(-|\mathbf{y}|^2\big)\big[f(\mathbf{x}+\mathbf{y}\sqrt{4kt}) - f(\mathbf{x})\big]\, d\mathbf{y}$$

since $\pi^{-3/2}\iiint e^{-|y|^2} f(\mathbf{x})d\mathbf{y} = f(\mathbf{x})\pi^{-3/2}\iiint e^{-|y|^2}\, d\mathbf{y} = f(\mathbf{x})$. Suppose now that $f$ satisfies

$$|f(\mathbf{x}) - f(\mathbf{y})| \le K|\mathbf{x}-\mathbf{y}| \tag{10.4}$$

Then we can estimate

$$|u(\mathbf{x},t) - f(\mathbf{x})| \le \frac{1}{(\pi)^{3/2}} \iiint \exp\big(-|\mathbf{y}|^2\big) K|\mathbf{y}|\sqrt{4kt}d\mathbf{y} = K\frac{\sqrt{4kt}}{\pi^{3/2}} \iiint \exp\big(-|\mathbf{y}|^2\big)|\mathbf{y}|\, d\mathbf{y} = C\sqrt{t}.$$

(Here the integral is easily seen to be finite, e.g. by introducing spherical coordinates.)
When $t\to 0$ the right hand side tends to 0 so (10.3) follows. As far as the assumption (10.4). We have to make some assumption on $f$, say that its continuously differentiable. If in addition $f$ has compact support or if the derivatives of $f$ are uniformly bounded then (10.4) holds. However, its also possibly to get by with weaker assumptions.

**Ex** Find the solution to the diffusion equation with initial data $f(x,y,z)=xy^2z$:

$$\begin{aligned}
u(x_0,y_0,z_0,t) &= \frac{1}{(4\pi kt)^{3/2}}\iiint \exp\Big(-\frac{(x-x_0)^2+(y-y_0)^2+(z-z_0)^2}{4kt}\Big)xy^2z\,dxdydz\\
&= \frac{1}{(4\pi kt)^{3/2}}\iiint \exp\Big(-\frac{x^2+y^2+z^2}{4kt}\Big)(x+x_0)(y+y_0)^2(z+z_0)\,dxdydz\\
&= \frac{1}{(4\pi kt)^{3/2}}\int e^{-x^2/4kt}(x+x_0)\,dx\int e^{-y^2/4kt}(y+y_0)^2\,dy\int e^{-z^2/4kt}(z+z_0)\,dz\\
&= \frac{1}{(4\pi kt)^{3/2}}\int e^{-x^2/4kt}x_0\,dx\int e^{-y^2/4kt}(y^2+y_0^2)\,dy\int e^{-z^2/4kt}z_0\,dz
\end{aligned}$$

since the integral of odd powers cancel $\int e^{-x^2/4kt}x^{2k+1}\,dx=0$. Moreover integrate by parts gives

$$\int e^{-y^2/4kt}y^2\,dy=\int \frac{y}{2kt}e^{-y^2/4kt}\,2kty\,dy=-\int e^{-y^2/4kt}2kt\,dy=-2kt\int e^{-y^2/4kt}\,dy$$

Hence

$$u(x_0,y_0,z_0,t)=x_0(y_0^2-2kt)z_0\frac{1}{(4\pi kt)^{3/2}}\int e^{-x^2/4kt}\,dx\int e^{-y^2/4kt}\,dy\int e^{-z^2/4kt}\,dz=x_0(y_0^2-2kt)z_0$$

**Schrödinger equation.** We claim that the fundamental solution for the Schrödinger equation in three space dimensions:

$$u_t=i\triangle u=i\big(u_{xx}+u_{yy}+u_{zz}\big),\qquad u(\mathbf{x},0)=f(\mathbf{x}), \tag{10.5}$$

is given by

$$u(\mathbf{x},t)=\frac{1}{(4\pi it)^{3/2}}\iiint \exp\Big(-\frac{|\mathbf{x}-\mathbf{y}|^2}{4it}\Big)f(\mathbf{y})\,d\mathbf{y}. \tag{10.7}$$

Formally this is obtained by replacing $k$ in (10.1)-(10.2) by the imaginary number $i$. However we must make a choice of square root $i^{1/2}$ in (10.6). For either possible choice $(i+1)/\sqrt{2}$ or $-(i+1)/\sqrt{2}$, (10.6) will satisfy the Schrödinger equation (10.4), but we must choose the square root in the right half plane $(i+1)/\sqrt{2}$ in order for the initial condition to be satisfied in (10.5). We will now see why the initial conditions are satisfied if we make this choice. We know that for $k$ real and positive:

$$\frac{1}{(4\pi kt)^{3/2}}\iiint \exp\Big(-\frac{|\mathbf{y}|^2}{4kt}\Big)\,d\mathbf{y}=1 \tag{10.7}$$

The integral however converges absolutely for $\operatorname{Re}k>0$. Since both sides of (10.6) are analytic functions for $\operatorname{Re}k>0$ if we choose the square root with positive real part, they must be equal. Passing to the limit $k=i+\varepsilon$, $\varepsilon\to 0$, $\varepsilon>0$, we get (10.7) with the square root defined with positive real part.