Lecture 11: 6.1 Laplace equation. Laplace equation:

$$\triangle u = 0. \tag{11.1}$$

A function satisfying Laplace equation is called harmonic.
The inhomogeneous version of Laplace equation is usually called Poisson's equation:

$$\triangle u = f. \tag{11.2}$$

Dirichlet problem in a domain $D \subset \mathbf{R}^n$ with boundary ∂D so to, given f and h, find u satisfying

$$\triangle u = f, \quad \text{in } D, \qquad u = h, \quad \text{on } \partial D. \tag{11.3}$$

Maximum Principle Let D be a connected bounded open set. Let u be a harmonic function in D which is continuous in $\overline{D} = D \cup \partial D$. Then the maximum and minimum values of u are attained on the boundary of D and nowhere inside (unless u is constant):

$$\min_{\mathbf{y}\in\partial D} u(\mathbf{y}) \leq u(\mathbf{x}) \leq \max_{\mathbf{y}\in\partial D} u(\mathbf{y}), \qquad \text{for all } \mathbf{x} \in D.$$

Proof The idea of the maximum principle is as follows, in two dimensions, say. At a maximum point inside we must have $u_x = u_y = 0$ and $u_{xx} \leq 0$ and $u_{yy} \leq 0$. At most maximum points we would have $u_{xx} < 0$ and $u_{yy} < 0$, which would contradict Laplace equation $u_{xx} + u_{yy} = 0$. However, it is possible that $u_{xx} = u_{yy} = 0$ at a maximum point, and we have to do something circumvent that possibility: Let $v(\mathbf{x}) = u(\mathbf{x}) + \varepsilon|\mathbf{x}|^2$, where $\varepsilon > 0$. Then, in two dimensions say,

$$\triangle v = \triangle u + \varepsilon \triangle (x^2 + y^2) = 0 + 4\varepsilon > 0$$

But $\triangle v \leq 0$ at an interior maximum point so v has no interior maximum. Since v is continuous in the closed compact set $\overline{D}$ it attains a maximum on $\overline{D}$ which must be on the boundary $\mathbf{x}_0 \in \partial D$. Then for all $\mathbf{x} \in D$:

$$u(\mathbf{x}) \leq v(\mathbf{x}) \leq v(\mathbf{x}_0) = u(\mathbf{x}_0) + \varepsilon|\mathbf{x}_0|^2 \leq \max_{\mathbf{y}\in\partial D} u(\mathbf{y}) + \varepsilon \max_{\mathbf{y}\in\partial D} |\mathbf{y}|^2$$

When $\varepsilon \to 0$ we get

$$u(\mathbf{x}) \leq \max_{\mathbf{y}\in\partial D} u(\mathbf{y}), \qquad \text{for all } \mathbf{x} \in D.$$

Uniqueness of the Dirichlet problem To prove uniqueness of solutions of (11.3) let us suppose that we have two solutions u and v satisfying (11.3). Then their difference w satisfies

$$\triangle w = 0, \quad \text{in } D, \qquad w = 0, \quad \text{on } \partial D.$$

But by the maximum principle it then follows that $0 \leq w(\mathbf{x}) \leq 0$, which means that $0 = w = u - v$.

Polar coordinates in two dimensions We want to express the Laplacian in two dimension in polar coordinates:

$$x = r\cos\theta, \qquad y = r\sin\theta.$$

The change of variable matrix is

$$\begin{bmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{bmatrix},$$

and hence

$$\begin{bmatrix} \dfrac{\partial r}{\partial x} & \dfrac{\partial r}{\partial y} \\ \dfrac{\partial \theta}{\partial x} & \dfrac{\partial \theta}{\partial y} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{bmatrix}^{-1} = \frac{1}{r}\begin{bmatrix} r\cos\theta & r\sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}. \tag{11.4}$$

That the derivative matrix of the inverse is the inverse of the derivative matrix, (11.4), follows by multiplying them together and using the chain rule:

$$\begin{bmatrix} \dfrac{\partial r}{\partial x} & \dfrac{\partial r}{\partial y} \\ \dfrac{\partial \theta}{\partial x} & \dfrac{\partial \theta}{\partial y} \end{bmatrix}\begin{bmatrix} \dfrac{\partial x}{\partial r} & \dfrac{\partial x}{\partial \theta} \\ \dfrac{\partial y}{\partial r} & \dfrac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} \dfrac{\partial r}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial r}{\partial y}\dfrac{\partial y}{\partial r} & \dfrac{\partial r}{\partial x}\dfrac{\partial x}{\partial \theta} + \dfrac{\partial r}{\partial y}\dfrac{\partial y}{\partial \theta} \\ \dfrac{\partial \theta}{\partial x}\dfrac{\partial x}{\partial r} + \dfrac{\partial \theta}{\partial y}\dfrac{\partial y}{\partial r} & \dfrac{\partial \theta}{\partial x}\dfrac{\partial x}{\partial \theta} + \dfrac{\partial \theta}{\partial y}\dfrac{\partial y}{\partial \theta} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

since e.g.

$$1 = \frac{\partial r}{\partial r} = \frac{\partial}{\partial r} r\big(x(r,\theta), y(r,\theta)\big) = \frac{\partial r}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial r}{\partial y}\frac{\partial y}{\partial r},$$

and

$$0 = \frac{\partial r}{\partial \theta} = \frac{\partial}{\partial \theta} r\big(x(r,\theta), y(r,\theta)\big) = \frac{\partial r}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial r}{\partial y}\frac{\partial y}{\partial \theta}.$$

One can of course alternatively derive the matrix in the left of (11.4) by solving for $r = (x^2 + y^2)^{1/2}$ and $\theta = \tan^{-1}(y/x)$ and differentiating.

By the chain rule and (11.4)

$$\begin{aligned} \frac{\partial}{\partial x} &= \frac{\partial r}{\partial x}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial x}\frac{\partial}{\partial \theta} = \cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}, \\ \frac{\partial}{\partial y} &= \frac{\partial r}{\partial y}\frac{\partial}{\partial r} + \frac{\partial \theta}{\partial y}\frac{\partial}{\partial \theta} = \sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}. \end{aligned}$$

Hence

$$\begin{aligned} \frac{\partial^2}{\partial x^2} = \Big(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\Big)\Big(\cos\theta\frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial \theta}\Big) &= \cos^2\theta\frac{\partial^2}{\partial r^2} + \frac{\sin^2\theta}{r^2}\frac{\partial^2}{\partial \theta^2} - \frac{2\cos\theta\sin\theta}{r}\frac{\partial^2}{\partial r\partial\theta} \\ &\quad + \frac{\cos\theta\sin\theta}{r^2}\frac{\partial}{\partial \theta} + \frac{\sin^2\theta}{r}\frac{\partial}{\partial r} + \frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial \theta}, \end{aligned}$$

and

$$\begin{aligned} \frac{\partial^2}{\partial y^2} = \Big(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\Big)\Big(\sin\theta\frac{\partial}{\partial r} + \frac{\cos\theta}{r}\frac{\partial}{\partial \theta}\Big) &= \sin^2\theta\frac{\partial^2}{\partial r^2} + \frac{\cos^2\theta}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{2\cos\theta\sin\theta}{r}\frac{\partial^2}{\partial r\partial\theta} \\ &\quad - \frac{\cos\theta\sin\theta}{r^2}\frac{\partial}{\partial \theta} + \frac{\cos^2\theta}{r}\frac{\partial}{\partial r} - \frac{\sin\theta\cos\theta}{r^2}\frac{\partial}{\partial \theta}. \end{aligned}$$

and therefore

$$\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}. \tag{11.5}$$

Cylindrical coordinates in three dimensions The cylindrical coordinates are given by

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad z = z. \tag{11.6}$$

The Laplacian in cylindrical coordinates is

$$\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}. \tag{11.7}$$

Spherical coordinates in three dimensions The spherical coordinates are obtained from the cylindrical coordinates by further introducing polar coordinates in the (z, r) plane for fixed θ as well:

$$z = \rho\cos\phi, \qquad r = \rho\sin\phi, \qquad \theta = \theta. \tag{11.8}$$

Then $\rho^2 = z^2 + r^2 = z^2 + x^2 + y^2$ is the square of the distance from the origin. The spherical coordinates are given by

$$x = \rho\sin\phi\cos\theta, \qquad y = \rho\sin\phi\sin\theta, \qquad z = \rho\cos\phi. \tag{11.9}$$

We now start with the cylindrical coordinates and make the change of variable $(r, z, \theta) \to (\rho, \phi, \theta)$ to spherical coordinates given by (11.8). Then we must for θ fixed change coordinates in (11.7). However, since we use polar coordinates in the (z, r) plane we can use the formula for the Laplacian in polar coordinates to write:

$$\frac{\partial^2}{\partial z^2} + \frac{\partial^2}{\partial r^2} = \frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} \tag{11.10}$$

In (11.7) we also have to deal with the term (again we think of θ as fixed and use the formulas from the polar coordinates)

$$\frac{1}{r}\frac{\partial}{\partial r} = \frac{1}{r}\Big(\frac{\partial \rho}{\partial r}\frac{\partial}{\partial \rho} + \frac{\partial \phi}{\partial r}\frac{\partial}{\partial \phi}\Big) = \frac{1}{r}\Big(\sin\phi\frac{\partial}{\partial \rho} + \frac{\cos\phi}{\rho}\frac{\partial}{\partial \phi}\Big) = \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\cos\phi}{\sin\phi}\frac{\partial}{\partial \phi}. \tag{11.11}$$

Using (11.10)-(11.11) in (11.7) we get

$$\triangle = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial \rho^2} + \frac{2}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2\sin^2\phi}\frac{\partial^2}{\partial \theta^2} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \frac{1}{\rho^2}\frac{\cos\phi}{\sin\phi}\frac{\partial}{\partial \phi}. \tag{11.12}$$

This can also be rewritten in divergence form

$$\triangle = \frac{1}{\rho^2\sin\phi}\Big(\frac{\partial}{\partial \rho}\rho^2\sin\phi\frac{\partial}{\partial \rho} + \frac{\partial}{\partial \theta}\frac{1}{\sin\phi}\frac{\partial}{\partial \theta} + \frac{\partial}{\partial \phi}\sin\phi\frac{\partial}{\partial \phi}\Big), \tag{11.13}$$

which is useful for applications.

We remark that there alternative notation. The textbook has switched the role of ϕ and θ. Moreover, once we derived the coordinates we will let r denote was is called ρ above, i.e. $\sqrt{x^2 + y^2 + z^2}$.