

Lecture 12: Poisson's formula. We will now derive a formula for the solution of the Dirichlet problem for the unit disc:

$$u_{xx} + u_{yy} = 0, \quad \text{for } x^2 + y^2 < 1, \quad u = h(\theta), \quad \text{when } x^2 + y^2 = 1,$$

and θ is the angular variable in polar coordinates in the plane. It is natural to express the Laplacian and $u(r, \theta)$ in polar coordinates:

$$(12.1) \quad u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0, \quad 0 \leq r < 1, \quad u(1, \theta) = h(\theta).$$

Assume that $u(r, \theta)$ for fixed r can be expanded in a Fourier series with coefficients depending on r :

$$(12.2) \quad u(r, \theta) = \sum_{k=-\infty}^{+\infty} R_k(r)e^{ik\theta}.$$

This has to satisfy (12.1) so we get

$$\sum R_k''(r)e^{ik\theta} + R_k'(r)e^{ik\theta} - \frac{k^2}{r^2}R_k e^{ik\theta} = 0,$$

which is satisfied if

$$R_k''(r) + \frac{2}{r}R_k' - \frac{k^2}{r^2}R_k(r) = 0.$$

The solution are $R_k(k) = c_k r^{|k|}$ (or $r^{-|k|}$ which is excluded since its singular when $r = 0$). Here $r^0 = 1$. Hence the general solution to (12.1) is

$$(12.3) \quad u(r, \theta) = \sum_{k=-\infty}^{+\infty} c_k r^{|k|} e^{ik\theta}.$$

We must now satisfy the boundary condition when $r = 1$ in (12.1):

$$(12.4) \quad h(\theta) = \sum_{k=-\infty}^{+\infty} c_k e^{ik\theta},$$

i.e. we must be able to expand the boundary values in a Fourier series. Assuming that we can do this we must have

$$(12.5) \quad c_n = \frac{1}{2\pi} \int_0^{2\pi} h(\phi) e^{-in\phi} d\phi.$$

In fact, if we multiply both sides of (12.4) with $e^{-in\theta}$ and integrate we get

$$\int_0^{2\pi} h(\theta) e^{-in\theta} d\theta = \sum_{k=-\infty}^{+\infty} c_k \int_0^{2\pi} e^{i(k-n)\theta} d\theta = c_n 2\pi,$$

since

$$(12.6) \quad \int_0^{2\pi} e^{im\theta} d\theta = e^{im\theta} \frac{1}{im} \Big|_0^{2\pi} = 0, \quad \text{if } m \neq 0, \quad \text{and} \quad \int_0^{2\pi} e^0 d\theta = 2\pi.$$

Let us assume that we can write h in terms of its Fourier series (12.4), with the coefficients satisfying (12.5). Using (12.5) in (12.3):

$$(12.7) \quad u(r, \theta) = \sum_{k=-\infty}^{+\infty} \frac{1}{2\pi} \int_0^{2\pi} h(\phi) e^{-ik\phi} d\phi r^{|k|} e^{ik\theta} = \frac{1}{2\pi} \int_0^{2\pi} \sum_{k=-\infty}^{+\infty} r^{|k|} e^{ik(\theta-\phi)} h(\phi) d\phi.$$

If we also change variables we get

$$u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \sum_{k=-\infty}^{+\infty} r^{|k|} e^{ik\phi} h(\theta + \phi) d\phi.$$

Here

$$(12.8) \quad \sum_{k=-\infty}^{+\infty} r^{|k|} e^{ik\phi} = \sum_{k=0}^{\infty} r^k e^{ik\phi} + \sum_{k=0}^{\infty} r^k e^{-ik\phi} - 1.$$

These are two geometric series in the complex variables $z = re^{i(\theta-\phi)}$ and $\bar{z} = re^{-i(\theta-\phi)}$. We have

$$(12.9) \quad \sum_{k=0}^{\infty} z^k = \frac{1}{1-z}, \quad \text{if } |z| < 1.$$

(In fact, $(1-z) \sum_{k=0}^n z^k = \sum_{k=0}^n z^k - \sum_{k=0}^n z^{k+1} = 1 - z^{n+1}$, and (12.9) follows when $n \rightarrow \infty$.)

Using (12.9) in (12.8) gives

$$(12.10) \quad \begin{aligned} \sum_{k=-\infty}^{+\infty} r^{|k|} e^{ik\phi} &= \frac{1}{1-re^{i\phi}} + \frac{1}{1-re^{-i\phi}} - 1 = \frac{2-r(e^{i\phi}+e^{-i\phi})}{1+r^2-r(e^{i\phi}+e^{-i\phi})} - \frac{1+r^2-r(e^{i\phi}+e^{-i\phi})}{1+r^2-r(e^{i\phi}+e^{-i\phi})} \\ &= \frac{1-r^2}{1+r^2-2r\cos\phi} = \frac{(1-r)(1+r)}{(1-r)^2+2r(1-\cos\phi)} = \frac{(1-r)(1+r)}{(1-r)^2+4r\sin^2(\phi/2)}. \end{aligned}$$

Using (12.10) in (12.7) we have hence obtained Poisson's formula for the solution of (12.1):

$$(12.11) \quad u(r, \theta) = \frac{1}{2\pi} \int_0^{2\pi} \frac{1-r^2}{1+r^2-2r\cos(\theta-\phi)} h(\phi) d\phi = \frac{1}{2\pi} \int_0^{2\pi} \frac{(1-r)(1+r)}{(1-r)^2+4r\sin^2(\phi/2)} h(\theta+\phi) d\phi.$$

The proof relied on that $h(\theta)$ can be expanded in a Fourier series. However, once we have the formula (12.11) we can directly prove that it satisfies (12.1). First its easy to check that it satisfies Laplace equation. This follows from the construction of the kernel as a convergent sum (12.10) of functions that satisfy Laplace equation in the interior, but can alternatively be check by direct computation. That (12.11) converges to the boundary values:

$$(12.12) \quad u(r, \theta) \rightarrow h(\theta), \quad \text{as } r \rightarrow 1,$$

require bit of work to prove but its well worth it since at the same time we would have shown that

$$(12.13) \quad \sum_{k=-\infty}^{+\infty} c_k e^{ik\theta} = \lim_{r \rightarrow 1} \sum_{k=-\infty}^{+\infty} c_k r^{|k|} e^{ik\theta} = \lim_{r \rightarrow 1} u(r, \theta) = h(\theta),$$

provided that $\sum |c_k| < \infty$. (12.13) is exactly the fact that the Fourier series converges to $h(\theta)$.

In (12.11), $\sin^2(\phi/2)$ only vanishes when $\phi = 0$ in the interval $[-\pi, \pi]$. The integrand in (12.11)

$$(12.14) \quad \frac{1}{2\pi} \frac{(1-r)(1+r)}{(1-r)^2 + 4r \sin^2(\phi/2)} \rightarrow \begin{cases} 0, & \text{when } \phi \neq 0 \\ \infty, & \text{when } \phi = 0 \end{cases}, \quad \text{as } r \rightarrow 1,$$

in such a way that, in view of (12.10) and (12.6),

$$(12.15) \quad \frac{1}{2\pi} \int_0^{2\pi} \frac{(1-r)(1+r)}{(1-r)^2 + 4r \sin^2(\phi/2)} d\phi = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sum_{k=-\infty}^{+\infty} r^{|k|} e^{ik\phi} d\phi = 1.$$

In other words the integrand tend to zero everywhere apart from at the origin where it tends to infinity, in such a way that the total integral is one. The limit is delta function in distribution theory.

We will now make two changes of variables which will enable us to take the limit of the integrand. First, we make the change of variables $s = 2 \sin(\phi/2)$, $ds = \cos(\phi/2)d\phi = 2^{-1}(4-s^2)^{1/2} d\phi$ and $\phi/2 = \sin^{-1}(s/2)$ so:

$$u(r, \theta) = \frac{1}{2\pi} \int_{-2}^2 \frac{(1-r)(1+r)}{(1-r)^2 + rs^2} h(\theta + 2 \sin^{-1}(s/2)) \frac{2ds}{\sqrt{4-s^2}}.$$

To understand better what is going we make another change of variables $t = (1-r)s$, to obtain

$$u(r, \theta) = \frac{1}{2\pi} \int_{-2/a}^{2/a} \frac{(1+r)}{1+rt^2} h(\theta + 2 \sin^{-1}(at/2)) \frac{2dt}{\sqrt{4-a^2t^2}}, \quad a = 1-r.$$

At least formally, if h is smooth we can now take the limit $r \rightarrow 1$, and get since $\sin^{-1} 0 = 0$;

$$u(r, \theta) \rightarrow \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2}{1+t^2} h(\theta) dt = h(\theta) \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{dt}{1+t^2} = h(\theta) \frac{1}{\pi} \tan^{-1} t \Big|_{-\infty}^{+\infty} = h(\theta) \frac{1}{\pi} \left(\frac{\pi}{2} - -\frac{\pi}{2} \right) = h(\theta).$$

A more careful analysis using that if h is differentiable then

$$|h(\theta + \phi) - h(\theta)| \leq K|\phi|,$$

can be used to justify this limit.