

Lecture 16: 12.1 Distributions. Let $C_0^\infty(\mathbf{R}^n)$ (or \mathcal{D}) denote the set of functions that are differentiable infinitely many times and that have compact support, i.e. that vanish for large $|\mathbf{x}|$.

The support of a continuous function ϕ is defined by

$$\text{supp } \phi = \{\mathbf{x} \in \mathbf{R}^n; \phi(\mathbf{x}) \neq 0\}.$$

A set K is called compact if it is closed and bounded.

We define the seminorms of all derivatives up to order N by

$$\|\phi\|_N = \sup_{\mathbf{x}} \sum_{|\alpha| \leq N} |\partial^\alpha \phi(\mathbf{x})|,$$

where $\alpha = (\alpha_1, \dots, \alpha_n)$ is a multi-index of length $|\alpha| = \alpha_1 + \dots + \alpha_n$ and $\partial^\alpha = \partial_{x_1}^{\alpha_1} \dots \partial_{x_n}^{\alpha_n}$.

We say that $\phi_n \rightarrow \phi$ in C_0^∞ if ϕ and all ϕ_j 's are supported in a fixed compact set K and $\|\phi_n - \phi\|_N \rightarrow 0$, as $n \rightarrow \infty$ for every fixed N .

A map $L: C_0^\infty \rightarrow \mathbf{R}$ is called a functional, i.e. for each function $\phi \in C_0^\infty$, $L(\phi)$ is a complex number.

A map $L: C_0^\infty \rightarrow \mathbf{R}$ is called linear if $L(a\phi + b\psi) = aL(\phi) + bL(\psi)$ for any constants a, b .

A map $L: C_0^\infty \rightarrow \mathbf{R}$ is called continuous if $\phi_n \rightarrow \phi$ implies that $L(\phi_n) \rightarrow L(\phi)$.

Definition. A **distribution** f is a continuous linear functional: $C_0^\infty \rightarrow \mathbf{R}$. We will write $\langle f, \phi \rangle$ for the corresponding real number.

A **bounded continuous function** can be viewed as a distribution given by $\langle f, \phi \rangle = \int f\phi dx$.

Even if f is a distribution we will sometimes use $\int f\phi dx$ to denote $\langle f, \phi \rangle$, keeping in mind that it is to be interpreted as a linear functional and not as a usual integral.

The delta function is the simplest example of a distribution which is not a function. The delta function at a , $\delta_a(x) = \delta(x - a)$ defined by

$$\langle \delta_a, \phi \rangle = \int \delta(x - a) \phi(x) dx = \phi(a).$$

The delta function can be thought of as a point mass: $\delta(x) = 0$, for $x \neq 0$, but $\int \delta(x) dx = 1$.

We have encountered the need of a delta function as a limit: The solution to the diffusion equation

$$u_t - k u_{xx} = 0, \quad u(x, 0) = f(x).$$

was given by

$$u(x, t) = \int_{-\infty}^{+\infty} S(x - y, t) f(y) dy, \quad \text{where} \quad S(x, t) = \frac{1}{\sqrt{4k\pi t}} e^{-|x|^2/4kt}.$$

When $t \rightarrow 0$ the integral tend to $f(x)$, i.e. $S(x - y, t) \rightarrow \delta(x - y)$, when $t \rightarrow 0$. Note how $S(x, t)$ peaks at $x = 0$ when $t \rightarrow 0$. One can think of the delta function as this limit.

A third interpretation in one variable is as the derivative of the step function: The step function H by $H(x) = 1$ for $x > 0$ and $H(x) = 0$ for $x < 0$. Then in the sense of distributions $H'(x) = \delta(x)$. In fact, if we formally integrate by parts we get

$$\int H'(x) \phi(x) dx = - \int H(x) \phi'(x) dx = - \int_0^\infty \phi'(x) dx = \phi(0), \quad \phi \in C_0^\infty.$$

Any derivative of a function is a distribution even if the function is not differentiable in the usual sense. In fact, one of the the main motivations to introduce distributions is to generalize the concept of derivative. We define the derivative by

$$\int (\partial^\alpha f) \phi dx \equiv (-1)^{|\alpha|} \int f \partial^\alpha \phi dx.$$

This defines a distribution that agrees with the usual derivative if f is smooth by integrating by parts. Any weak limit of a distribution is a distribution. We say that $f_n \rightarrow f$ weakly if

$$\int f_n \phi dx \rightarrow \int f \phi dx, \quad \phi \in C_0^\infty.$$

Moreover, it follows directly from the definitions that $\partial^\alpha f_n \rightarrow \partial^\alpha f$ if $f_n \rightarrow f$.

We define the convolution of tow functions by

$$\phi * f(\mathbf{x}) = \int \phi(\mathbf{x} - \mathbf{y}) f(\mathbf{y}) d\mathbf{y} = \int \phi(\mathbf{y}) f(\mathbf{x} - \mathbf{y}) d\mathbf{y}$$

Let ϕ be a smooth function going to zero fast as $|\mathbf{x}| \rightarrow \infty$, and such that $\int \phi(\mathbf{x}) d\mathbf{x} = 1$.

Set $\phi_\varepsilon(\mathbf{x}) = \phi(\mathbf{x}/\varepsilon)/\varepsilon^n$. Then $\int \phi_\varepsilon(\mathbf{x}) d\mathbf{x} = 1$. $\phi_\varepsilon(\mathbf{x}) \rightarrow \delta(\mathbf{x})$, as $\varepsilon \rightarrow 0$ and $\phi_\varepsilon * f(\mathbf{x}) \rightarrow f(\mathbf{x})$. In fact

$$\phi_\varepsilon * f(\mathbf{x}) = \int \phi(\mathbf{y}/\varepsilon) f(\mathbf{x} - \mathbf{y}) d\mathbf{y}/\varepsilon^n = \int \phi(\mathbf{z}) f(\mathbf{x} - \varepsilon\mathbf{z}) d\mathbf{z} \rightarrow \int \phi(\mathbf{z}) f(\mathbf{x}) d\mathbf{z} = f(\mathbf{x}), \quad \text{as } \varepsilon \rightarrow 0,$$

if f is a continuous differentiable function with bounded derivative $|f(\mathbf{x} + \mathbf{y}) - f(\mathbf{x})| \leq K|\mathbf{y}|$, since

$$\phi_\varepsilon * f(\mathbf{x}) - f(\mathbf{x}) = \int \phi(\mathbf{z})(f(\mathbf{x} - \varepsilon\mathbf{z}) - f(\mathbf{x})) d\mathbf{z} \leq \int |\phi(\mathbf{z})| K\varepsilon |\mathbf{z}| d\mathbf{z} = C\varepsilon.$$

Any distribution f is the weak limit of a sequence of $f_n \in C_0^\infty$. In fact, with $\phi_\varepsilon(x) = \phi(x/\varepsilon)/\varepsilon^n$ as above, the convolution $f_k(x) = \phi_{1/k} * f(x) = \int f(y)\phi_{1/k}(x - y) dy = \langle f, \phi_{1/k}(x - \cdot) \rangle$ is well defined as the distribution f acting on the test function $\phi_{1/k}(x - y)$, considered as a function of y . That $f_k \in C^\infty$ is seen by looking at difference quotients $(f_k(x + he_j) - f_k(x))/h$ and using that $(\phi_{1/k}(x + he_j) - \phi_{1/k}(x))/h \rightarrow \partial_j \phi_{1/k}(x)$ in C_0^∞ . To show that $f_k \rightarrow f$:

$$\langle f_k, \psi \rangle = \iint f(y)\phi_{1/k}(x - y) dy \psi(x) dx = \int f(y) \int \phi_{1/k}(x - y)\psi(x) dx dy = \langle f, \phi_{1/k} * \psi \rangle$$

It follows from the above that $\phi_{1/k} * \psi \rightarrow \psi$, and hence $f_k \rightarrow f$ weakly.

Problem Let $\psi(t) = e^{-1/t}$, when $t > 0$, and $\psi(t) = 0$, when $t \leq 0$. Show that $\psi \in C^\infty(\mathbf{R})$. Let $\eta(x) = \psi(1 - |x|^2)$, where $|x| = \sqrt{x_1^2 + \dots + x_n^2}$. Show that $\eta(x) \in C_0^\infty(\mathbf{R}^n)$.