

Lecture 18: 12.2: The fundamental solution for the wave equation. Let us now derive the fundamental solution for the wave operator:

$$\square E = \delta(t, x), \quad \square = \partial_t^2 - \Delta, \quad (t, x) \in \mathbf{R}^{1+n}$$

is not hard to derive from the symmetries as well. Since \square is invariant under Lorentz transformations we expect the fundamental solution $E(t, x)$ to be invariant under Lorentz transformations as well, which means that it should be of the form $E(t, x) = f(t^2 - |x|^2)$, where f is a distribution. Plugging this into the equation gives after some calculation

$$(18.1) \quad 4\rho f''(\rho) + 2(1+n)f'(\rho) = 0, \quad \rho = t^2 - |x|^2$$

when $(t, x) \neq (0, 0)$. This has the solution

$$(18.2) \quad \begin{cases} f(\rho) = c_1 H(\rho), & \text{if } n = 1 \\ f(\rho) = c_2 H(\rho) \rho^{-1/2}, & \text{if } n = 2 \\ f(\rho) = c_3 \delta(\rho), & \text{if } n = 3 \end{cases}$$

The constants can be calculated in the same way as we did for the fundamental solution of Δ .

$$(18.3) \quad \begin{cases} E(t, x) = c_1 H(t - |x|), & \text{if } n = 1 \\ E(t, x) = c_2 H(t - |x|) (t^2 - |x|^2)^{-1/2}, & \text{if } n = 2 \\ E(t, x) = c_3 \delta(t^2 - |x|^2) H(t), & \text{if } n = 3 \end{cases}$$

Problem: Show in each case that (18.2) is a solution of (18.1).

Problem: If $n = 3$ prove that

$$(18.4) \quad \delta(t^2 - |x|^2) H(t) = \delta(t - |x|) / 2|x|$$

Problem Prove that $E(t, x)$ given above are fundamental solutions of \square and find the constants c_n .

Using the fundamental solution E for \square we can now solve the Cauchy problem

$$\begin{aligned} \square u(t, x) &= F \\ u(0, x) &= f(x), \quad u_t(0, x) = g(x) \end{aligned}$$

In fact let $u_0(t, x) = u(t, x)H(t)$ and $F_0(t, x) = F(t, x) = H(t)$ then

$$\begin{aligned} \square u_0(t, x) &= \square u(t, x)H(t) = (\square u(t, x))H(t) + 2u_t(t, x)\delta(t) + u(t, x)\delta'(t) \\ &= F(t, x)H(t) + 2u_t(0, x)\delta(t) + u(0, x)\delta'(t) - u_t(0, x)\delta(t) = F_0(t, x) + g(x)\delta(t) + f(x)\delta'(t) \end{aligned}$$

and hence

$$u_0(t, x) = E * (F_0(t, x) + g(x)\delta(t) + f(x)\delta'(t)) = E * F_0 + E * (g(x)\delta(t)) + \partial_t E * (f(x)\delta(t))$$

Let us now derive the solution formula if $n = 3$ in which case $E(t, x) = \delta(t - |x|) / 4\pi|x|$ and hence

Problem Show that

$$(18.5) \quad E * F_0(t, x) = \int \int F_0(t - s, x - y) \delta(s - |y|) \frac{1}{4\pi|y|} dy ds = \int_{|y| \leq t} \frac{F_0(t - |y|, x - y)}{4\pi|y|} dy$$

and that

$$E * (g(x)\delta(t)) = t \int_{\omega \in S^2} \frac{g(x - t\omega)}{4\pi} dS(\omega)$$

where $dS(\omega)$ is the surface measure on the sphere S^2 .