Lecture 19: 12.3 The Fourier transform. The Fourier transform $\mathcal{F}: f \to \hat{f}$ is defined to be

$$\hat{f}(\xi) = \int_{\mathbf{R}^n} f(x)\, e^{-ix\cdot\xi}\, dx. \tag{19.1}$$

The Fourier transform is invertible, in fact we will prove Fourier's inversion formula:

$$f(x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \hat{f}(\xi)\, e^{ix\cdot\xi}\, dx \tag{19.2}$$

The Fourier transform makes sense for a very general class of functions and even distributions. However, it is natural to first define it for a more restrictive class and afterwards extend the definition by continuity. This is the Schwartz class $\mathcal{S}$ consisting of all infinitely differentiable functions that are rapidly decreasing:

$$\sup_x |x^\beta \partial^\alpha \phi(x)| < \infty$$

for all multi-indices α and β. The importance of this class is that $\mathcal{F}: \mathcal{S} \to \mathcal{S}$. which follows from the following identities for the Fourier transform:

$$\mathcal{F}: \partial_j f(x) \to i\xi_j \hat{f}(\xi), \qquad \mathcal{F}: x_j f(x) \to i\partial_j \hat{f}(\xi) \tag{19.3}$$

(19.3) follows from integrating by parts in (19.1) respectively differentiating below the integral sign. That $\mathcal{F}: \mathcal{S} \to \mathcal{S}$ now follows using (19.1) and integrating by parts

$$\xi^\alpha \partial_\xi^\beta \hat{\phi}(\xi) = \int \xi^\alpha (-i)^{|\beta|} x^\beta e^{-ix\cdot\xi} \phi(x)\, dx = (-1)^{|\alpha|}(-i)^{|\alpha|} \int \partial_x^\alpha \big(x^\beta \phi(x)\big)\, dx$$

which can be bounded by $\sup_x \big|\partial_x^\alpha\big(x^\beta\phi(x)\big)\big|(1+|x|)^{1+n} \int (1+|x|)^{-1-n} dx \leq C$, since $\phi \in \mathcal{S}$.

Note also that by changing variables we get two more simple properties

$$\mathcal{F}: f(ax) \to a^{-n}\hat{f}(\xi/a), \qquad \mathcal{F}: f(x+h) \to \hat{f}(\xi)e^{ih\cdot\xi} \tag{19.4}$$

The proof of (3.2) uses:

Lemma.

$$\mathcal{F}: e^{-|ax|^2/2} \to (2\pi)^{n/2} a^{-n} e^{-|\xi/a|^2/2} \tag{19.5}$$

Proof. Let $\phi(x) = e^{-|x|^2/2}$. Since $(\partial_j + x_j)\phi(x) = 0$ it follows from (4) that $(i\xi_j + i\partial_j)\hat{\phi}(\xi) = 0$, $j = 1, ..., n$. This differential equation has the solution $\hat{\phi}(\xi) = c_n e^{-|\xi|^2/2}$. In fact, if we integrate it for $j = 1$ we get $\hat{\phi}(\xi) = F_2(\xi_2, \cdots, \xi_n)e^{-\xi_1^2/2}$. Plugging this into the same equation gives $(i\xi_k + i\partial_k)F_2(\xi_2, \cdot, \xi_n) = 0$, for $k \geq 2$. Integrating this equation for $k = 2$ gives $F_2(\xi_2, \cdots, \xi_n) = F_3(\xi_3, \cdots, \xi_n)e^{-\xi_2^2/2}$. Repeating this gives that $\hat{\phi}(\xi) = c_n e^{-\xi_1^2/2} \cdots e^{-\xi_n^2/2} = c_n e^{-|\xi|^2/2}$. It therefore only remains to calculate c_n. However, by (3.1) $c_n = \hat{\phi}(0) = \int e^{-|x|^2/2}\, dx$. If $n = 2$ this integral can easily be calculated by introducing polar coordinates $\int_{\mathbf{R}^2} e^{-|x|^2/2}\, dx = 2\pi \int_0^\infty e^{-r^2/2}\, r\, dr = 2\pi$. In general we can write $\int_{\mathbf{R}^n} e^{-|x|^2/2}\, dx = (\int_{\mathbf{R}} e^{-x_1^2/2}\, dx_1)^n$ and $\int_{\mathbf{R}} e^{-x_1^2/2}\, dx_1 = (\int_{\mathbf{R}^2} e^{-|x|^2/2}\, dx)^{1/2}$ so $c_n = (2\pi)^{n/2}$. $\square$

Lemma. *If* $\phi \in \mathcal{S}$ *set* $\phi_\varepsilon(x) = \phi(x/\varepsilon)/\varepsilon^n$, *then*

$$\int f(x)\,\phi_\varepsilon(x)\,dx = \int f(\varepsilon x)\,\phi(x)\,dx \to f(0)\int \phi(x)\,dx \quad \varepsilon \to 0, \quad \textit{for} \quad f \in \mathcal{S}$$

Proof. Since $|f(\varepsilon x)\,\phi(x)| \le \sup_y |f(y)|\,|\phi(x)|$ the lemma follows from the theorem of Dominated converge. It is also easy to prove directly; since

$$|f(\varepsilon x)\,\phi(x) - f(0)\phi(x)| \le \varepsilon \sup_y ||y| f'(y)|\,|\phi(x)|$$

the difference of the two integrals is bounded by $C\varepsilon$. □

We also have

$$\int \hat{\phi}\psi\,d\xi = \int \phi\hat{\psi}dx, \quad \phi,\psi \in \mathcal{S} \tag{19.6}$$

In fact, both sides of (19.6) are equal to the double integral

$$\iint \phi(x)\psi(\xi)\,e^{-ix\cdot\xi}\,dx d\xi$$

It follows from using (19.3) that it suffices to prove (19.2) for $x = 0$ since its translation invariant. Using (19.6) gives

$$\int \hat{\phi}(x)\ \ f(\varepsilon x)\,dx = \int \phi(\varepsilon\xi)\,\hat{f}(\xi)\,d\xi$$

By Lemma 2 we get as $\varepsilon \to 0$

$$\int \hat{\phi}(x)\,dx\,f(0) = \phi(0)\int \hat{f}(\xi)\,d\xi$$

Picking $\phi(x) = e^{-|x|^2/2}$ we get from Lemma 1 and its proof that $\int \hat{\phi}(x)\,dx = (2\pi)^n$ and Fourier's inversion formula (19.2) follows. Using Fourier's inversion formula and (19.6) we get Parseval's formula

$$\int \phi(x)\overline{\psi(x)}\,dx = \frac{1}{(2\pi)^n}\int \hat{\phi}(\xi)\overline{\hat{\psi}(\xi)}\,d\xi \tag{19.7}$$

In particular;

$$\int |\phi(x)|^2\,dx = \frac{1}{(2\pi)^n}\int |\hat{\phi}(\xi)|^2\,d\xi.$$

We can therefore extend the Fourier transform by continuity to a map $\mathcal{F}: L^2 \to L^2$. (This follows since for any $f \in L^2$ one can find a sequence of functions $f_n \in \mathcal{S}$ such that $f_n \to f$ in L^2 and it follows that $\hat{f}_n \in \mathcal{S}$ converges in L^2 to some function $\hat{f}$.)

Problem Find the Fourier transform of $e^{-|x|}$, $x \in \mathbf{R}$.

Problem Find the inverse Fourier transform of $\sin|\xi|/|\xi|$, $\xi \in \mathbf{R}$.

The Fourier transform of a distribution is defined through duality using (19.6)

$$< \widehat{f}, \phi > = < f, \widehat{\phi} >$$

In order for this to be well defined for all ϕ we must assume that f is a tempered distribution, i.e. an element of the dual space $\mathcal{S}'$, of $\mathcal{S}$ or a continuous linear functional on $\mathcal{S}$ with respect to the seminorms $\rho_{\alpha,\beta}(\phi) = \sup_x |x^\beta \partial^\alpha \phi(x)|$, since $\mathcal{F}: \mathcal{S} \to \mathcal{S}$. We can hence extend the Fourier transform to a map $\mathcal{F}: \mathcal{S}' \to \mathcal{S}'$. If f has compact support then $\hat{f}$ is the smooth function $\langle f(x), e^{-ix\cdot\xi}\rangle$. (A distribution has compact support in K if $\langle f,\phi\rangle = 0$, $\operatorname{supp}\phi \cap K = \emptyset$.) We have $\mathcal{F}\colon \delta_a(x) \to e^{-ia\cdot\xi}$

Problem Compute the Fourier transform of $e^{-ax^2/2}$, for Re $a \ge 0$.

Problem Compute the Fourier transform of the function $f = 1$.