Lecture 2: 1.2 Transport Equation, 2.1 Wave equation, 2.2 Energy Conservation.

1.2 The Initial Value Problem for the Transport Equation.

$$(\partial_t + c\partial_x)u(x,t) = u_t(x,t) + cu_x(x,t) = 0 \tag{2.1}$$

This equation just says that u is constant in the direction $(1, c)$, i.e. u is constant along the *characteristic lines* $x - ct = \eta$. In fact

$$\frac{d}{dt}u(ct+\eta,t) = u_t(ct+\eta,t) + cu_x(ct+\eta,t) = 0,$$

if (2.1) holds. It follows that the general solution to (2.1) is

$$u(x,t) = f(\eta) = f(x-ct) \tag{2.2}$$

for some function f. The solution is determined uniquely by posing the initial condition

$$u(x,0) = f(x) \tag{2.3}$$

The solution is a wave being transported at a speed c.
Remark If we in general try to solve

$$au_t + bu_x = 0, \qquad u(x,0) = f(x)$$

we see that it only works if $a \neq 0$, i.e. if the problem is *non-characteristic*. If $a = 0$ and $b \neq 0$ then the first equation says that $u_x = 0$ which contradicts the second equation unless $f'(x) = 0$.

2.1 The Initial Value Problem for the Wave Equation.

$$u_{tt} - c^2 u_{xx} = (\partial_t - c\partial_x)(\partial_t + c\partial_x)u = 0 \tag{2.4}$$

has the general solution

$$u(x,t) = v(x+ct) + w(x-ct) \tag{2.5}$$

for arbitrary functions v and w. Note that the solution at time t consist of two waves one traveling to the right and one traveling to the left, both with speed c. Its easy to check that (2.5) is a solution to (2.4) since $(\partial_t + c\partial_x)w(x-ct) = 0$ and $(\partial_t - c\partial_x)v(x+ct) = 0$. That it is the general solution follows by changing to *characteristic coordinates* $\xi = x + ct$ and $\eta = x - ct$. Then $x = \dfrac{\xi+\eta}{2}$, $t = \dfrac{\xi-\eta}{2c}$ so $\dfrac{\partial}{\partial\xi} = \dfrac{\partial x}{\partial\xi}\dfrac{\partial}{\partial x} + \dfrac{\partial t}{\partial\xi}\dfrac{\partial}{\partial t} = \dfrac{1}{2}\partial_x + \dfrac{1}{2c}\partial_t = \dfrac{1}{2c}(\partial_t + c\partial_x)$ and $\partial_\eta = \cdots = -\dfrac{1}{2c}(\partial_t - c\partial_x)$. Hence

$$\partial_\xi\partial_\eta u = -\frac{1}{4c^2}(\partial_t - c\partial_x)(\partial_t + c\partial_x)u = 0$$

Integration gives $u_\eta = h(\eta)$ and hence $u = w(\eta) + v(\xi)$ for any functions v and w.

There are two families of *characteristic lines* $x-ct=$constant and $x+ct=$constant. The solution (2.5) is the sum of two traveling waves, $v(x+ct)$ traveling to the left at speed c along the characteristic lines $x+ct$=constant and $w(x-ct)$ traveling to the right along the characteristic lines $x-ct$ =constant.

We claim that initial value problem for (2.4) with initial data

$$u(x,0) = f(x), \qquad u_t(0,x) = g(x) \tag{2.6}$$

has the solution

$$u(t,x) = \frac{1}{2}\big(f(x+ct) + f(x-ct)\big) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds. \tag{2.7}$$

In fact if we plug the expression (2.5) into (2.6) we get

$$f(x) = u(0,x) = v(x) + w(x), \qquad g(x) = u_t(0,x) = cv'(x) - cw'(x)$$

Integrating the second equation gives a system

$$f(x) = v(x) + w(x), \qquad G(x) = v(x) - w(x), \qquad \text{where} \qquad G(x) = \int g(x)\,dx$$

which has the solution

$$v(x) = \frac{1}{2}f(x) + \frac{1}{2c}G(x), \qquad w(x) = \frac{1}{2}f(x) - \frac{1}{2c}G(x).$$

Hence

$$u(x,t) = v(x+ct) + w(x-ct) = \frac{1}{2}\big(f(x+ct) + f(x-ct)\big) + \frac{1}{2c}\big(G(x+ct) - G(x-ct)\big)$$

from which (2.7) follows since $G(x+ct) - G(x-ct) = \int_{x-ct}^{x+ct} g(y)\,dy$. Alternatively, you can just check that (2.7) satisfy the initial conditions (2.6) and the equation (2.4) since it is of the form (2.5).

2.2 Causality:. Principle of Causality: No part of the wave can go faster than the speed of light. Changing the initial conditions at $(x_0, 0)$ can only influence the solution in a forward light cone $x_0 - ct \le x \le x_0 + ct$ as is seen from the solution formula (2.7). By the same argument:

$$f(x) = 0 \text{ and } g(x) = 0 \quad \text{when } |x| \ge R \quad \Longrightarrow \quad u(x,t) = 0, \quad \text{when } |x| \ge R + ct. \tag{2.8}$$

In addition; the solution (2.7) at a point (x_0, t_0) depends only on the initial values $f(x)$ and $g(x)$ in the interval $x_0 - ct \le x \le x_0 + ct$.

2.2 Energy. There is energy conservation for a solution of the wave equation. The total energy consists a kinetic and potential part:

$$E(t) = \int u_t(x,t)^2 + c^2 u_x(x,t)^2\,dx,$$

where the integral is from $-\infty$ to $+\infty$. To be sure that the integral converges we assume (2.8) holds. If we differentiate below the integral sign and integrate by parts we get

$$E'(t) = \int 2u_t u_{tt} + 2c^2 u_x u_{tx}\,dx = \int 2u_t u_{tt} - 2c^2 u_{xx} u_t\,dx = \int 2u_t(u_{tt} - c^2 u_{xx})\,dx = 0$$

u is a solution of the wave equation $u_{tt} - c^2 u_{xx} = 0$. Here we used that since u vanishes for large $|x|$:

$$0 = \int_{-\infty}^{+\infty} \partial_x(u_x u_t)\,dx = \int_{-\infty}^{+\infty} u_{xx} u_t + u_x u_{xt}\,dx \quad \Longrightarrow \quad \int_{-\infty}^{+\infty} u_x u_{xt}\,dx = -\int_{-\infty}^{+\infty} u_{xx} u_t\,dx.$$

Since $E'(t) = 0$ it follows that the energy is conserved:

$$E(t) = E(0). \tag{2.9}$$

Remark There is a version of the energy identity in a truncated light cone. For simplicity let us assume that $c=1$. Let $C=\{(x,t);\, |x-x_0|\le t_0-t\}$ be the backward light cone from (x_0,t_0) and, for some $\tau<t_0$, let $C_\tau=\{(t,x)\in C;\, 0\le t\le \tau\}$ be the truncated cone:

$$C_\tau=\{(x,t);\, -(t_0-t)\le x-x_0\le t_0-t,\, 0\le t\le\tau\}=\{(x,t);\, 0\le t\le \min\,(t_0-|x-x_0|,\tau)\}.$$

Here $$\min\,(t_0-|x-x_0|,\tau)=\begin{cases} t_0+x-x_0, & \text{when} \quad x_0-t_0\le x\le x_0-(t_0-\tau)\\ \tau, & \text{when} \quad x_0-(t_0-\tau)\le x\le x_0+(t_0-\tau)\\ t_0+x_0-x, & \text{when} \quad x_0+(t_0-\tau)\le x\le x_0+t_0\end{cases}.$$

The boundary of C_τ consist of a top $T=\{(x,\tau);\,|x-x_0|\le t_0-\tau\}$, a bottom $B=\{(x,0);\,|x-x_0|\le t_0\}$, and right and left sides of the cone $K_\pm=\{(t,x);\, x-x_0=\pm(t_0-t),\, 0\le t\le\tau\}$. Let the Energy be:

$$E(t)=\int_{|x-x_0|\le t_0-t} u_t(x,t)^2+u_x(x,t)^2\,dx,$$

and the Flux be

$$F(t)=\int_{t_0-t\le x_0-x\le t_0}\big((u_t+u_x)(x,t_0-x_0+x)\big)^2\,dx+\int_{t_0-t\le x-x_0\le t_0}\big((u_t-u_x)(x,t_0-x+x_0)\big)^2\,dx.$$

$E(\tau)$ is an integral over the top, $E(0)$ an integral over the bottom and F an integral over the sides. We claim that

$$E(\tau)+F(\tau)=E(0). \tag{2.10}$$

Since $F(\tau)\ge 0$ the energy inequality $E(\tau)\le E(0)$ follows. (2.10) follows from integrating

$$\partial_t\big(u_t^2+u_x^2\big)-2\partial_x\big(u_tu_x\big)=2u_t\big(u_{tt}-u_{xx}\big)=0$$

over C_τ:

$$0=\iint_{C_\tau}\partial_t\big(u_t^2+u_x^2\big)-2\partial_x\big(u_tu_x\big)\,dxdt.$$

Here by the fundamental theorem of calculus

$$\iint_{C_\tau}\partial_t\big(u_t^2+u_x^2\big)\,dx\,dt=\int_{x_0-t_0}^{x_0+t_0}\int_0^{\min\,(t_0-|x-x_0|,\tau)}\partial_t\big(u_t^2+u_x^2\big)\,dt\,dx$$

$$=\int_{x_0-t_0}^{x_0+t_0}\big(u_t^2+u_x^2\big)(x,\min\,(t_0-|x-x_0|,\tau))\,dx-\int_{x_0-t_0}^{x_0+t_0}\big(u_t^2+u_x^2\big)(x,0)\,dx$$

$$=\int_{x_0-t_0}^{x_0-(t_0-\tau)}\big(u_t^2+u_x^2\big)(x,t_0+x-x_0)\,dx+\int_{x_0-(t_0-\tau)}^{x_0+(t_0-\tau)}\big(u_t^2+u_x^2\big)(x,\tau)\,dx+\int_{x_0+(t_0-\tau)}^{x_0+t_0}\big(u_t^2+u_x^2\big)(x,t_0+x_0-x)\,dx$$

$$-E(0)=\int_{x_0-t_0}^{x_0-(t_0-\tau)}\big(u_t^2+u_x^2\big)(x,t_0+x-x_0)\,dx+E(\tau)+\int_{x_0+(t_0-\tau)}^{x_0+t_0}\big(u_t^2+u_x^2\big)(x,t_0+x_0-x)\,dx-E(0),$$

and

$$-2\iint_{C_\tau}\partial_x\big(u_tu_x\big)\,dx\,dt=-2\int_0^\tau\int_{x_0-(t_0-t)}^{x_0+t_0-t}\partial_x\big(u_tu_x\big)\,dx\,dt$$

$$=2\int_0^\tau\big(u_tu_x\big)\big(x_0-(t_0-t),t\big)\,dt-2\int_0^\tau\big(u_tu_x\big)\big(x_0+t_0-t,t\big)\,dt$$

$$=2\int_{x_0-t_0}^{x_0-(t_0-\tau)}\big(u_tu_x\big)(x,t_0+x-x_0)\,dx-2\int_{x_0+(t_0-\tau)}^{x_0+t_0}\big(u_tu_x\big)(x,t_0+x_0-x)\,dx.$$

Since $u_t^2+u_x^2+2u_tu_x=(u_t+u_x)^2$ and $u_t^2+u_x^2-2u_tu_x=(u_t-u_x)^2$ it follows that

$$0=\iint_{C_\tau}\partial_t\big(u_t^2+u_x^2\big)\,dx\,dt-2\iint_{C_\tau}\partial_x\big(u_tu_x\big)\,dx\,dt=F(\tau)+E(\tau)-E(0).$$