Lecture 20: 12.3 Fourier transform.

Lemma. *If $\phi \in \mathcal{S}$ set $\phi_\varepsilon(x) = \phi(x/\varepsilon)/\varepsilon^n$, then*

$$\int f(x)\,\phi_\varepsilon(x)\,dx = \int f(\varepsilon x)\,\phi(x)\,dx \to f(0)\int \phi(x)\,dx \quad \varepsilon \to 0, \quad \textit{for} \quad f \in \mathcal{S}$$

Proof. Since $|f(\varepsilon x)\,\phi(x)| \le \sup_y |f(y)|\,|\phi(x)|$ the lemma follows from the theorem of Dominated converge. It is also easy to prove directly; since

$$|f(\varepsilon x)\,\phi(x) - f(0)\phi(x)| \le \varepsilon \sup_y ||y| f'(y)|\,|\phi(x)|$$

the difference of the two integrals is bounded by $C\varepsilon$. □

We also have

$$\int \hat{\phi}\psi\,d\xi = \int \phi\hat{\psi}dx, \quad \phi, \psi \in \mathcal{S} \tag{20.1}$$

In fact, both sides of (20.1) are equal to the double integral

$$\iint \phi(x)\psi(\xi)\,e^{-ix\cdot\xi}\,dx d\xi$$

It follows from using (19.3) that it suffices to prove (19.2) for $x = 0$ since its translation invariant. Using (20.1) gives

$$\int \hat{\phi}(x)\;\; f(\varepsilon x)\,dx = \int \phi(\varepsilon\xi)\,\hat{f}(\xi)\,d\xi$$

By Lemma 2 we get as $\varepsilon \to 0$

$$\int \hat{\phi}(x)\,dx\,f(0) = \phi(0)\int \hat{f}(\xi)\,d\xi$$

Picking $\phi(x) = e^{-|x|^2/2}$ we get from Lemma 1 and its proof that $\int \hat{\phi}(x)\,dx = (2\pi)^n$ and Fourier's inversion formula (19.2) follows. Using Fourier's inversion formula and (20.1) we get Parseval's formula

$$\int \phi(x)\overline{\psi(x)}\,dx = \frac{1}{(2\pi)^n}\int \hat{\phi}(\xi)\overline{\hat{\psi}(\xi)}\,d\xi \tag{20.2}$$

In particular;

$$\int |\phi(x)|^2\,dx = \frac{1}{(2\pi)^n}\int |\hat{\phi}(\xi)|^2\,d\xi.$$

We can therefore extend the Fourier transform by continuity to a map $\mathcal{F} : L^2 \to L^2$. (This follows since for any $f \in L^2$ one can find a sequence of functions $f_n \in \mathcal{S}$ such that $f_n \to f$ in L^2 and it follows that $\hat{f}_n \in \mathcal{S}$ converges in L^2 to some function $\hat{f}$.)

Problem Find the Fourier transform of $e^{-|x|}$, $x \in \mathbf{R}$.

Problem Find the inverse Fourier transform of $\sin|\xi|/|\xi|$, $\xi \in \mathbf{R}$.

The Fourier transform of a distribution is defined through duality using (19.6)

$$< \widehat{f}, \phi > = < f, \widehat{\phi} >$$

In order for this to be well defined for all ϕ we must assume that f is a tempered distribution, i.e. an element of the dual space $\mathcal{S}'$, of $\mathcal{S}$ or a continuous linear functional on $\mathcal{S}$ with respect to the seminorms $\rho_{\alpha,\beta}(\phi) = \sup_x |x^\beta \partial^\alpha \phi(x)|$, since $\mathcal{F} : \mathcal{S} \to \mathcal{S}$. We can hence extend the Fourier transform to a map $\mathcal{F} : \mathcal{S}' \to \mathcal{S}'$. If f has compact support then $\hat{f}$ is the smooth function $\langle f(x), e^{-ix\cdot\xi}\rangle$. (A distribution has compact support in K if $\langle f, \phi\rangle = 0$, $\operatorname{supp}\phi \cap K = \emptyset$.) We have $\mathcal{F}\colon \delta_a(x) \to e^{-ia\cdot\xi}$

Problem Compute the Fourier transform of $e^{-ax^2/2}$, for Re $a \ge 0$.

Problem Compute the Fourier transform of the function $f = 1$.

12.4 Solving initial value problem with the Fourier transform. Recall that the Fourier transform is given by

$$\hat{f}(\xi) = \int f(x)e^{-ix\xi}\,dx$$

Let the convolution be defined by

$$K * g(x) = \int K(y)g(x-y)\,dy = \int K(x-y)g(y)\,dy$$

It is easy to see that

$$\mathcal{F}:\ f * g \to \hat{f}\hat{g} \tag{20.3}$$

The heat equation. Let us now look on the heat equation

$$\begin{aligned} \partial_t u(t,x) - \triangle u(t,x) &= 0 \\ u(0,x) &= g(x) \end{aligned}$$

taking the Fourier transform with respect to the space variables only:

$$\hat{u}(t,\xi) = \int u(t,x)\, e^{-ix\cdot\xi}\,dx$$

gives

$$\begin{aligned} \partial_t \hat{u}(t,\xi) + |\xi|^2 \hat{u}(t,\xi) &= 0 \\ \hat{u}(0,\xi) &= \hat{g}(\xi) \end{aligned}$$

Hence

$$\hat{u}(t,\xi) = e^{-t|\xi|^2}\hat{g}(\xi)$$

By Lemma 3.1 with $a = 1/\sqrt{2t}$

$$K_t(x) = \mathcal{F}^{-1}(e^{-t|\xi|^2})(x) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}, \qquad t > 0$$

and by (20.3)

$$u(t,x) = K_t * g(x) = \int \frac{1}{(4\pi t)^{n/2}} e^{-|x-y|^2/4t} g(y)\,dy, \qquad t > 0$$

Problem: Verify directly that $K_t * g(x) \to g(x)$, when $t \to 0$.