

Lecture 21: 12.4 Solving Initial Value Problems with the Fourier Transform. Recall that the Fourier transform is given by

$$\hat{f}(\xi) = \int f(x)e^{-ix\xi} dx$$

Let the convolution be defined by

$$K * g(x) = \int K(y)g(x-y) dy = \int K(x-y)g(y) dy$$

It is easy to see that

$$\mathcal{F} : f * g \rightarrow \hat{f}\hat{g}$$

The wave equation. Let us now look on the wave equation

$$(21.1) \quad \partial_t^2 u(t, x) - \Delta u(t, x) = 0$$

$$(21.2) \quad u(0, x) = f(x), \quad u_t(0, x) = g(x)$$

taking the Fourier transform $\hat{u}(t, \xi) = \int u(t, x) e^{-ix \cdot \xi} dx$ gives

$$\partial_t^2 \hat{u}(t, \xi) + |\xi|^2 \hat{u}(t, \xi) = 0$$

$$\hat{u}(0, \xi) = \hat{f}(\xi), \quad \partial_t \hat{u}(0, \xi) = \hat{g}(\xi)$$

It is easy to see that this second order ODE has the solution

$$\hat{u}(t, \xi) = \cos(t|\xi|) \hat{f}(\xi) + \frac{\sin(t|\xi|)}{|\xi|} \hat{g}(\xi)$$

The inverse Fourier transform of $\cos(t|\xi|)$ and $\sin(t|\xi|)/|\xi|$ are so far not defined since these functions are not in L^2 , in \mathbf{R}^n , if $n \geq 2$. not functions but distributions. In fact the inverse Fourier transform of these functions can not even be defined as a function. Instead we will in the next section define the inverse Fourier transform of these functions as distributions.

Problem If $\xi \in \mathbf{R}$ find the inverse Fourier transform of $\cos(t|\xi|) = \cos(t\xi) = (e^{it\xi} + e^{-it\xi})/2$ and $\sin(t|\xi|)/|\xi| = \sin(t\xi)/\xi$ and use it to obtain the following integral representation of the solution of (21.1)-(21.2):

$$u(t, x) = \frac{1}{2}(f(x+t) + f(x-t)) + \frac{1}{2} \int_{x-t}^{x+t} g(y) dy$$

Problem Derive the formula for the solution in three space dimensions, see book.

Problem Show that

$$|\partial_t \hat{u}(t, \xi)|^2 + |\xi|^2 |\hat{u}(t, \xi)|^2 = |\hat{g}(\xi)|^2 + |\xi|^2 |\hat{f}(\xi)|^2$$

and use it to prove the energy identity

$$\int |\partial_t u(t, x)|^2 + \sum_{i=1}^n |\partial_i u(t, x)|^2 dx = \int |g|^2 + \sum_{i=1}^n |\partial_i f|^2 dx$$