

Lecture 5: 9.1 Energy and causality of waves. We will study the wave equation in space time:

$$(5.1) \quad \partial_t^2 u - c^2 \Delta u = 0, \quad \text{where} \quad \Delta = \partial_x^2 + \partial_y^2 + \partial_z^2.$$

The characteristic cone Recall that in one space dimension the characteristic curves for the wave equation are $x - x_0 = \pm c(t - t_0)$. This is the path a particle traveling with the speed of light c will follow. If we rotate this around the $t = t_0$ axis we get the *characteristic cone or light cone*;

$$(5.2) \quad |\mathbf{x} - \mathbf{x}_0|^2 \equiv (x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = c^2(t - t_0)^2.$$

It is the union of all light rays from the point (\mathbf{x}_0, t_0) . The solid light cone is the inside; $|\mathbf{x} - \mathbf{x}_0| < c|t - t_0|$. The part with $t > t_0$ is called the future or forward light cone and the part with $t < t_0$ is called the past or backward light cone. At any fixed time the light cone is a sphere: $|\mathbf{x} - \mathbf{x}_0| = c|t_0 - t|$.

The divergence theorem states that if W is a volume bounded by a surface S with outward unit normal $\mathbf{n} = (n_1, n_2, n_3)$ and $\mathbf{F} = (F_1, F_2, F_3)$ is a continuously differentiable vector field in W then

$$(5.3) \quad \iiint_W \nabla \cdot \mathbf{F} \, dx dy dz = \iint_S \mathbf{F} \cdot \mathbf{n} \, dS, \quad \text{where}$$

$$(5.4) \quad \nabla \cdot \mathbf{F} = \frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} + \frac{\partial F_3}{\partial z} \quad \text{and} \quad \mathbf{F} \cdot \mathbf{n} = F_1 n_1 + F_2 n_2 + F_3 n_3.$$

A one dimensional analogue is the First Fundamental Theorem of Calculus:

$$(5.5) \quad f(b) - f(a) = \int_a^b f'(x) \, dx.$$

A two dimensional analogue says that if D is a region in the plane with boundary curve C then

$$\iint_D \left(\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y} \right) dx dy = \int_C F_1 n_1 + F_2 n_2 \, ds$$

where ds is the arc length. (This is in fact equivalent to Green's Theorem.)

Proof of the divergence theorem for convex sets. We say that a domain W is **convex** if for every two points in W the line segment between the two points is also in W , e.g. any sphere or rectangular box is convex. Since $\mathbf{F} = F_1 \mathbf{i} + F_2 \mathbf{j} + F_3 \mathbf{k}$ the theorem follows from proving it for each of the three vector fields $F_1 \mathbf{i}$, $F_2 \mathbf{j}$ and $F_3 \mathbf{k}$ separately. The theorem for the vector field $F_3 \mathbf{k}$ states that

$$(5.6) \quad \iiint_W \frac{\partial F_3}{\partial z} \, dx dy dz = \iint_S F_3 n_3 \, dS.$$

Since W is convex we can write $W = \{(x, y, z); f_1(x, y) \leq z \leq f_2(x, y), (x, y) \in D\}$. Its surface S has two parts $S_1 = \{(x, y, z); z = f_1(x, y), (x, y) \in D\}$, $S_2 = \{(x, y, z); z = f_2(x, y), (x, y) \in D\}$. We have

$$\begin{aligned} \iiint_W \frac{\partial F_3}{\partial z} \, dz dx dy &= \iint_D \int_{f_1(x,y)}^{f_2(x,y)} \frac{\partial F_3}{\partial z} \, dz \, dx dy \\ &= \iint_D F_3((x, y, f_2(x, y))) \, dx dy - \iint_D F_3((x, y, f_1(x, y))) \, dx dy, \end{aligned}$$

by (5.5). This is an integral over S and if we can prove that $n_3 dS = dx dy$ on S_2 and $n_3 dS = -dx dy$ on S_1 we are done. This follows since the surface area dS above a small rectangle $dx dy$ in D is $dx dy / n_3$, since the ratio is given by the dot product of the unit normals to D and S .

Conservation of energy. If we multiply the wave equation (5.1) by u_t we get

$$(5.7) \quad 0 = (u_{tt} - c^2 \Delta u)u_t = \frac{1}{2} \partial_t (u_t^2 + c^2 |\nabla u|^2) - c^2 \nabla \cdot (u_t \nabla u), \quad \text{or}$$

$$0 = (u_{tt} - c^2 (u_{xx} - u_{yy} - u_{zz}))u_t = \frac{1}{2} \partial_t (u_t^2 + c^2 (u_x^2 + u_y^2 + u_z^2)) - c^2 (\partial_x (u_t u_x) + \partial_y (u_t u_y) + \partial_z (u_t u_z)).$$

If we integrate over a ball of radius R : $B_R = \{\mathbf{x}; |\mathbf{x}| \leq R\}$ we get from the divergence theorem

$$0 = \iiint_{B_R} \frac{1}{2} \frac{\partial}{\partial t} (u_t^2 + |\nabla u|^2) - c^2 \nabla \cdot (u_t \nabla u) d\mathbf{x} = \frac{1}{2} \frac{d}{dt} \iiint_{B_R} (u_t^2 + |\nabla u|^2) d\mathbf{x} - c^2 \iint_{S_R} (u_t \nabla u) \cdot \mathbf{n} dS,$$

where $S_R = \{\mathbf{x}; |\mathbf{x}| = R\}$ is the sphere of radius R and \mathbf{n} is the outward unit normal to S_R . If we assume that u vanishes for large $|\mathbf{x}|$ the last integral will vanish for large R . If we let $R \rightarrow \infty$ we get

$$\frac{d}{dt} E(t) = 0, \quad \text{where} \quad E(t) \equiv \iiint (u_t^2 + c^2 |\nabla u|^2)(\mathbf{x}, t) d\mathbf{x}, \quad \text{so} \quad E(t) = E(0).$$

Principle of causality. Let us for simplicity of notation assume that $c=1$ and $\mathbf{x}_0 = \mathbf{0}$. There is a version of the energy identity in a truncated light cone:

$$C_\tau = \{(\mathbf{x}, t); |\mathbf{x}| \leq t_0 - t, 0 \leq t \leq \tau\} = \{(\mathbf{x}, t); 0 \leq t \leq \min(t_0 - |\mathbf{x}|, \tau)\}, \quad \text{where}$$

$$(5.8) \quad \min(t_0 - |\mathbf{x}|, \tau) = \begin{cases} \tau, & \text{when } |\mathbf{x}| \leq t_0 - \tau, \\ t_0 - |\mathbf{x}|, & \text{when } t_0 - \tau \leq |\mathbf{x}| \leq t_0. \end{cases}$$

The boundary of C_τ consist of a top $T = \{(\mathbf{x}, \tau); |\mathbf{x} - \mathbf{x}_0| \leq t_0 - \tau\}$, a bottom $B = \{(x, 0); |\mathbf{x}| \leq t_0\}$, and the side of the cone $K = \{(\mathbf{x}, t); |\mathbf{x}| = (t_0 - t), 0 \leq t \leq \tau\}$. Let the Energy be:

$$(5.9) \quad E(t) = \int_{|\mathbf{x}| \leq t_0 - t} u_t(\mathbf{x}, t)^2 + |\nabla u(\mathbf{x}, t)|^2 d\mathbf{x},$$

and the Flux be

$$(5.10) \quad F = \int_{t_0 - \tau \leq |\mathbf{x}| \leq t_0} (u_t^2 + |\nabla u|^2 - 2u_t u_r)(\mathbf{x}, t_0 - |\mathbf{x}|) d\mathbf{x},$$

where $u_r = \mathbf{n} \cdot \nabla u$, and $\mathbf{n} = \mathbf{x}/|\mathbf{x}|$ is the outward unit normal to the spheres $|\mathbf{x}| = \text{constant}$. $E(\tau)$ is an integral over the top, $E(0)$ an integral over the bottom and F an integral over side. We claim that

$$(5.11) \quad E(\tau) + F - E(0) = 0, \quad \text{and} \quad E(\tau) \leq E(0).$$

Here $F \geq 0$ since $u_t^2 + |\nabla u|^2 - 2u_t u_r = (u_t - u_r)^2 + |\nabla u|^2 - u_r^2 \geq 0$, since $|u_r| = |\mathbf{n} \cdot \nabla u| \leq |\mathbf{n}| |\nabla u| = |\nabla u|$.

To prove (5.11) we integrate both terms in (5.7) over C_τ . By the fundamental theorem of calculus

$$(5.12) \quad \iiint \iiint_{C_\tau} \partial_t (u_t^2 + |\nabla u|^2)(\mathbf{x}, t) d\mathbf{x} dt = \iiint \iiint_{|\mathbf{x}| \leq t_0} \int_0^{\min(t_0 - |\mathbf{x}|, \tau)} \partial_t (u_t^2 + |\nabla u|^2)(\mathbf{x}, t) dt d\mathbf{x} \\ = \iiint \iiint_{|\mathbf{x}| \leq t_0 - \tau} (u_t^2 + |\nabla u|^2)(\mathbf{x}, \tau) d\mathbf{x} + \iiint \iiint_{t_0 - \tau \leq |\mathbf{x}| \leq t_0} (u_t^2 + |\nabla u|^2)(\mathbf{x}, t_0 - |\mathbf{x}|) d\mathbf{x} - \iiint \iiint_{|\mathbf{x}| \leq t_0} (u_t^2 + |\nabla u|^2)(\mathbf{x}, 0) d\mathbf{x}.$$

This is $E(\tau)$ plus part of F minus $E(0)$. By the divergence theorem on the balls of radius $t_0 - t$:

$$\iiint \iiint_{C_\tau} \nabla \cdot (u_t \nabla u)(\mathbf{x}, t) d\mathbf{x} dt = \int_0^\tau \iiint \iiint_{|\mathbf{x}| \leq t_0 - t} \nabla \cdot (u_t \nabla u)(\mathbf{x}, t) d\mathbf{x} dt = \int_0^\tau \iint_{|\mathbf{x}| = t_0 - t} \mathbf{n} \cdot (u_t \nabla u)(\mathbf{x}, t) dS(\mathbf{x}) dt,$$

where dS is the surface area element on the spheres of radius $t_0 - t$, and if we change variables $r = t_0 - t$;

$$(5.13) \quad = \int_{t_0 - \tau}^{t_0} \iint_{|\mathbf{x}| = r} \mathbf{n} \cdot (u_t \nabla u)(\mathbf{x}, t_0 - r) dS(\mathbf{x}) dr = \iiint \iiint_{t_0 - \tau \leq |\mathbf{x}| \leq t_0} \mathbf{n} \cdot (u_t \nabla u)(\mathbf{x}, t_0 - |\mathbf{x}|) d\mathbf{x}.$$

If we subtract of 2 times (5.13) from (5.12) and use that the integral of (5.7) is 0 we get (5.11).

One could also have proven (5.11) using the four dimensional divergence theorem applied to the integral of (5.7) over C_τ , which is the proof in Strauss.