Lecture 6: 3.2 Reflection of waves. We now want to solve the initial value problem for the wave equation on a half line:

$$v_{tt} - v_{xx} = 0, \qquad 0 < x < \infty, \qquad v(x,0) = f(x), \quad v_t(x,0) = g(x), \tag{6.1}$$

with Dirichlet boundary condition:

$$v(0,t) = 0.$$

In order for the problem to be solvable we will assume that the initial conditions are compatible with the boundary conditions, i.e. $f(0) = 0$ and $g(0) = 0$. Then general solution can be written

$$v(x,t) = F_0(t+x) + G_0(x-t),$$

where in order to satisfy the boundary condition we must have $v(0,t) = F_0(t) + G_0(-t) = 0$ so $G(-t) = -F_0(t)$. Therefore the general solution can be written

$$v(x,t) = F_0(t+x) - F_0(t-x), \tag{6.2}$$

for some function $F_0(x)$ defined for $-\infty < x < +\infty$. To satisfy the initial conditions we must have

$$\begin{cases} F_0(x) - F_0(-x) &= f(x), \\ F_0'(x) - F_0'(-x) &= g(x) \end{cases} \Leftrightarrow \begin{cases} F_0(x) - F_0(-x) &= f(x), \\ F_0(x) + F_0(-x) &= G(x) \end{cases}, \qquad \text{where} \quad G(x) = \int g(x)\, dx.$$

Hence

$$F_0(x) = \frac{1}{2}\big(G(x) + f(x)\big), \qquad F_0(-x) = \frac{1}{2}\big(G(x) - f(x)\big), \tag{6.3}$$

If $0 < x < t$ it follows from (6.2) and the first part of (6.3) that

$$v(x,t) = \frac{1}{2}\big(f(t+x) - f(t-x)\big) + \frac{1}{2}\big(G(t+x) - G(t-x)\big) = \frac{1}{2}\big(f(t+x) - f(t-x)\big) + \frac{1}{2}\int_{t-x}^{t+x} g(y)\, dy.$$

If $x > t$ it follows from (6.2) and the first and second parts of (6.3) that

$$v(x,t) = \frac{1}{2}\big(f(t+x) + f(x-t)\big) + \frac{1}{2}\big(G(t+x) - G(x-t)\big) = \frac{1}{2}\big(f(t+x) + f(x-t)\big) + \frac{1}{2}\int_{x-t}^{x+t} g(y)\, dy.$$

Spherically symmetric solutions to the Wave equation in space-time.
Let $r = |\mathbf{x}| = \sqrt{x_1^2 + x_2^2 + x_3^2}$ and $\partial_i = \dfrac{\partial}{\partial x^i}$.
Then $\partial_i r = \dfrac{x_i}{r}$. In fact, $2r\partial_i r = \partial_i r^2 = \partial_i\big(x_1^2 + x_2^2 + x_3^2\big) = 2x_i$, and diving by $2r$ gives the result.

$$\triangle f(|x|) = \sum_j \partial_j \partial_j f(|x|) = \sum_j \partial_j \Big(f'(|x|)\frac{x_j}{|x|}\Big) = \sum_j f''(|x|)\frac{x_j^2}{|x|^2} + f'(|x|)\Big(\frac{1}{|x|} - \frac{x_j^2}{|x|^3}\Big),$$

and hence

$$\triangle f(r) = f''(r) + \frac{2}{r}f'(r) = \frac{1}{r}\frac{d^2}{dr^2}\big(rf(r)\big) = \frac{1}{r^2}\frac{d}{dr}\Big(r^2 f'(r)\Big), \qquad r = |x| \neq 0.$$

We now want to solve the wave equation in 3 space dimensions

$$u_{tt} - \triangle u = 0, \qquad u(\mathbf{x},0) = f(\mathbf{x}), \qquad u_t(\mathbf{x},0) = g(\mathbf{x}).$$

It turns out that it simpler to consider the special case of spherically symmetric data first, i.e. data depending only on $|\mathbf{x}|$; $f(\mathbf{x}) = \overline{f}(|\mathbf{x}|)$ and $g(\mathbf{x}) = \overline{g}(|\mathbf{x}|)$. Then we expect the solution to only depend on $|\mathbf{x}|$ and t: $u(\mathbf{x},t) = \overline{u}(|\mathbf{x}|,t)$. We will show that in fact we can find a solution of this form. If we plug this ansatz into the equation we get

$$\overline{u}_{tt}(r,t) - \frac{1}{r}\frac{\partial^2}{\partial r^2}\big(r\overline{u}(r,t)\big) = 0, \qquad \overline{u}(r,0) = \overline{f}(r), \qquad \overline{u}_t(r,0) = \overline{g}(r). \tag{6.4}$$

Hence we get a one dimensional wave equation for $v(r,t) = r\overline{u}(r,t)$ with boundary condition:

$$v_{tt} - v_{rr} = 0, \quad r > 0, \qquad v(r,0) = r\overline{f}(r) = v_0(r), \quad v_t(r,0) = r\overline{g}(r) = v_1(r), \qquad v(0,t) = 0.$$

By the formulas from the previous section, then general solution looks like

$$\overline{u}(r,t) = \frac{v(r,t)}{r} = \frac{F_0(t+r) - F_0(t-r)}{r}.$$

If $0 < r < t$ then the solution of the initial value problem is given by

$$\overline{u}(r,t) = \frac{v(r,t)}{r} = \frac{1}{2r}\big(v_0(t+r) - v_0(t-r)\big) + \frac{1}{2r}\int_{t-r}^{t+r} v_1(y)\,dy, \tag{6.5}$$

and if $r > t$ then

$$\overline{u}(r,t) = \frac{v(r,t)}{r} = \frac{1}{2r}\big(v_0(t+r) + v_0(r-t)\big) + \frac{1}{2r}\int_{r-t}^{r+t} v_1(y)\,dy.$$

9.2 The wave equation in space time. We want to solve the wave equation in 3 space dimensions

$$u_{tt} - \triangle u = 0, \qquad u(\mathbf{x},0) = f(\mathbf{x}), \qquad u_t(\mathbf{x},0) = g(\mathbf{x}). \tag{6.6}$$

It turns out that we can get a formula from using the formula for the spherically symmetric case:

$$u(\mathbf{x}_0,t_0) = \mathcal{S}g(\mathbf{x}_0,t_0) + \frac{\partial}{\partial t_0}\mathcal{S}f(\mathbf{x}_0,t_0), \qquad \text{where} \quad \mathcal{S}g(\mathbf{x}_0,t_0) \equiv \frac{1}{4\pi t_0}\iint_{|\mathbf{x}-\mathbf{x}_0|=t_0} g(\mathbf{x})\,dS, \tag{6.7}$$

where dS is the surface measure on the sphere of radius t_0 centered at $\mathbf{x}_0$. Note that the sphere $|\mathbf{x}-\mathbf{x}_0| = t_0$ is the intersection of the backward light cone from $(\mathbf{x}_0,t_0)$ with the initial surface where $t=0$. (6.7) is called *Kirchoff's formula.* We have already discussed how to reduce the proof to the case when $f\equiv 0$; Since $v = \mathcal{S}f$ satisfies the wave equation $v_{tt} - v_{xx} = 0$ with initial data $v(x,0) = 0$ and $v_t(x,0) = f(x)$ it follows that $w = v_t$ satisfies the wave equation $w_{tt} - w_{xx} = 0$ with initial data $w(x,0) = v_t(0,x) = f(x)$ and $w_t(x,0) = v_{tt}(x,0) = v_{xx}(x,0) = 0$.
To prove (6.7), let $\overline{u}(r,t)$ be the average of $u(\mathbf{x},t)$ over the sphere $\{|\mathbf{x}|=r\}$ of radius r centered at $\mathbf{0}$:

$$\overline{u}(r,t) = \frac{1}{4\pi r^2}\iint_{|\mathbf{x}|=r} u(\mathbf{x},t)\,dS = \frac{1}{4\pi}\int_0^\pi\int_0^{2\pi} u(\mathbf{x},t)\,\sin\phi\,d\theta\,d\phi \tag{6.8}$$

where $(x,y,z) = (r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\theta)$ are the spherical coordinates. The Laplacian in spherical coordinates is

$$\triangle\, u = \frac{1}{r^2\sin\phi}\Big[\frac{\partial}{\partial r}\Big(\sin\phi\; r^2\frac{\partial u}{\partial r}\Big) + \frac{\partial}{\partial\theta}\Big(\frac{1}{\sin\phi}\frac{\partial u}{\partial\theta}\Big) + \frac{\partial}{\partial\phi}\Big(\frac{\partial u}{\partial\phi}\Big)\Big], \tag{6.9}$$

see section 6.1 for a proof. Now

$$\triangle\,\overline{u}(r) = \frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial}{\partial r}\overline{u}(r)\Big) = \frac{1}{4\pi}\int_0^\pi\int_0^{2\pi}\frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial}{\partial r}u(\mathbf{x})\Big)\,\sin\phi\,d\theta\,d\phi = \frac{1}{4\pi}\int_0^\pi\int_0^{2\pi}\triangle u(\mathbf{x})\,\sin\phi\,d\theta\,d\phi,$$

since using (6.9) the angular derivatives integrate out to 0:

$$\begin{aligned} r^2\int_0^\pi\int_0^{2\pi}\Big(\triangle u(\mathbf{x}) - \frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial}{\partial r}u(\mathbf{x})\Big)\Big)\,\sin\phi\,d\theta\,d\phi &= \int_0^\pi\int_0^{2\pi}\Big[\frac{\partial}{\partial\theta}\Big(\frac{1}{\sin\phi}\frac{\partial u}{\partial\theta}\Big) + \frac{\partial}{\partial\phi}\Big(\frac{\partial u}{\partial\phi}\Big)\Big]\,d\theta\,d\phi \\ &= \int_0^\pi\frac{1}{\sin\phi}\frac{\partial u}{\partial\theta}\Big|_{\theta=0}^{2\pi}d\phi + \int_0^{2\pi}\frac{\partial u}{\partial\phi}\Big|_{\phi=0}^{\pi}d\theta = 0+0. \end{aligned}$$

since $\frac{\partial u}{\partial\theta}\Big|_{\theta=0}^{2\pi} = 0$ and $\frac{\partial u}{\partial\phi}(r,\theta,\phi) = -\frac{\partial u}{\partial\phi}(r,\theta+\pi,\phi)$, if $\phi=0$ or $\phi=\pi$.

Hence $\overline{u}(r,t)$ satisfies the radial wave equation

$$\overline{u}_{tt} - \overline{u}_{rr} - \frac{2}{r}\overline{u}_r = \overline{u}_{tt} - \triangle\overline{u} = \frac{1}{4\pi}\int_0^\pi\int_0^{2\pi}\big(u_{tt}(\mathbf{x},t) - \triangle u(\mathbf{x},t)\big)\,\sin\phi\,d\theta\,d\phi = 0$$

and the solution when $r<t$ is represented by (6.5). In particular, if we assume that $f=0$ and let $r\to 0$ we get since the average over the sphere tends to the value in the middle

$$u(\mathbf{0},t) = \lim_{r\to 0}\overline{u}(r,t) = \lim_{r\to 0}\frac{1}{2r}\int_{t-r}^{t+r} y g(y)\,dy = t\overline{g}(t) = \frac{1}{4\pi t}\iint_{|\mathbf{x}|=t} g(\mathbf{x})\,dS.$$

This proves (6.7) in case $\mathbf{x}_0 = 0$ and the general case follows from this by translation: Use the above formula on $w(\mathbf{x},t) = u(\mathbf{x}+\mathbf{x}_0,t)$. Then $w(\mathbf{0},t) = u(\mathbf{x}_0,t)$, $w(\mathbf{x},0) = 0$, $\partial_t w(\mathbf{x},0) = g(\mathbf{x}+\mathbf{x}_0)$.