Lecture 7: 9.2 The wave equation in space time. We want to solve the initial value problem for the wave equation in 3 space dimensions:

$$u_{tt} - \triangle u = 0, \qquad u(\mathbf{x},0) = f(\mathbf{x}), \qquad u_t(\mathbf{x},0) = g(\mathbf{x}). \tag{7.1}$$

It turns out that we can get a formula from using the formula for the spherically symmetric case:

$$u(\mathbf{x}_0,t_0) = \mathcal{S}g(\mathbf{x}_0,t_0) + \frac{\partial}{\partial t_0}\mathcal{S}f(\mathbf{x}_0,t_0), \qquad \text{where} \quad \mathcal{S}g(\mathbf{x}_0,t_0) \equiv \frac{1}{4\pi t_0}\iint_{|\mathbf{x}-\mathbf{x}_0|=t_0} g(\mathbf{x})\,dS, \tag{7.2}$$

where dS is the surface measure on the sphere of radius t_0 centered at $\mathbf{x}_0$. Note that the sphere $|\mathbf{x}-\mathbf{x}_0| = t_0$ is the intersection of the backward light cone from $(\mathbf{x}_0,t_0)$ with the initial surface where $t = 0$. (7.2) is called *Kirchoff's formula.* We have already discussed how to reduce the proof to the case when $f \equiv 0$; Since $v = \mathcal{S}f$ satisfies the wave equation $v_{tt} - v_{xx} = 0$ with initial data $v(x,0) = 0$ and $v_t(x,0) = f(x)$ it follows that $w = v_t$ satisfies the wave equation $w_{tt} - w_{xx} = 0$ with initial data $w(x,0) = v_t(0,x) = f(x)$ and $w_t(x,0) = v_{tt}(x,0) = v_{xx}(x,0) = 0$.
To prove (7.2), let $\overline{u}(r,t)$ be the average of $u(\mathbf{x},t)$ over the sphere $\{|\mathbf{x}|=r\}$ of radius r centered at $\mathbf{0}$:

$$\overline{u}(r,t) = \frac{1}{4\pi r^2}\iint_{|\mathbf{x}|=r} u(\mathbf{x},t)\,dS = \frac{1}{4\pi}\int_0^\pi\int_0^{2\pi} u(\mathbf{x},t)\,\sin\phi\,d\theta\,d\phi \tag{7.3}$$

where $(x,y,z) = (r\cos\theta\sin\phi, r\sin\theta\sin\phi, r\cos\theta)$ are the spherical coordinates. In fact the surface area element is $dS = r\sin\phi d\theta\; r d\phi$. The Laplacian in spherical coordinates is

$$\triangle\, u = \frac{1}{r^2\sin\phi}\Big[\frac{\partial}{\partial r}\Big(\sin\phi\; r^2\frac{\partial u}{\partial r}\Big) + \frac{\partial}{\partial\theta}\Big(\frac{1}{\sin\phi}\frac{\partial u}{\partial\theta}\Big) + \frac{\partial}{\partial\phi}\Big(\frac{\partial u}{\partial\phi}\Big)\Big], \tag{7.4}$$

see section 6.1 for a proof. Now

$$\triangle\,\overline{u}(r) = \frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial}{\partial r}\overline{u}(r)\Big) = \frac{1}{4\pi}\int_0^\pi\int_0^{2\pi}\frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial}{\partial r}u(\mathbf{x})\Big)\,\sin\phi\,d\theta\,d\phi = \frac{1}{4\pi}\int_0^\pi\int_0^{2\pi}\triangle u(\mathbf{x})\,\sin\phi\,d\theta\,d\phi,$$

since using (7.4) the angular derivatives integrate out to 0:

$$\begin{aligned} r^2\int_0^\pi\int_0^{2\pi}\Big(\triangle u(\mathbf{x}) - \frac{1}{r^2}\frac{\partial}{\partial r}\Big(r^2\frac{\partial}{\partial r}u(\mathbf{x})\Big)\Big)\,\sin\phi\,d\theta\,d\phi &= \int_0^\pi\int_0^{2\pi}\Big[\frac{\partial}{\partial\theta}\Big(\frac{1}{\sin\phi}\frac{\partial u}{\partial\theta}\Big) + \frac{\partial}{\partial\phi}\Big(\frac{\partial u}{\partial\phi}\Big)\Big]\,d\theta\,d\phi \\ &= \int_0^\pi \frac{1}{\sin\phi}\frac{\partial u}{\partial\theta}\Big|_{\theta=0}^{2\pi}\,d\phi + \int_0^{2\pi}\frac{\partial u}{\partial\phi}\Big|_{\phi=0}^{\pi}\,d\theta = 0+0. \end{aligned}$$

since $\displaystyle\frac{\partial u}{\partial\theta}\Big|_{\theta=0}^{2\pi} = 0$ and $\displaystyle\frac{\partial u}{\partial\phi}(r,\theta,\phi) = -\frac{\partial u}{\partial\phi}(r,\theta+\pi,\phi)$, if $\phi = 0$ or $\phi = \pi$.

Hence $\overline{u}(r,t)$ satisfies the radial wave equation

$$\overline{u}_{tt} - \overline{u}_{rr} - \frac{2}{r}\overline{u}_r = \overline{u}_{tt} - \triangle\overline{u} = \frac{1}{4\pi}\int_0^\pi\int_0^{2\pi}\big(u_{tt}(\mathbf{x},t) - \triangle u(\mathbf{x},t)\big)\,\sin\phi\,d\theta\,d\phi = 0$$

and the solution when $r < t$ is represented by (6.5). In particular, if we assume that $f = 0$ and let $r \to 0$ we get since the average over the sphere tends to the value in the middle

$$u(\mathbf{0},t) = \lim_{r\to 0}\overline{u}(r,t) = \lim_{r\to 0}\frac{1}{2r}\int_{t-r}^{t+r} y g(y)\,dy = t\overline{g}(t) = \frac{1}{4\pi t}\iint_{|\mathbf{x}|=t} g(\mathbf{x})\,dS.$$

This proves (7.2) in case $\mathbf{x}_0 = 0$ and the general case follows from this by translation: Use the above formula on $w(\mathbf{x},t) = u(\mathbf{x}+\mathbf{x}_0,t)$. Then $w(\mathbf{0},t) = u(\mathbf{x}_0,t)$, $w(\mathbf{x},0) = 0$, $\partial_t w(\mathbf{x},0) = g(\mathbf{x}+\mathbf{x}_0)$.

9.2 The wave equation in two space dimensions. We now want to solve the initial value problem for the wave equation in two space dimensions:

$$u_{tt} - u_{xx} - u_{yy} = 0, \qquad u(x,y,0) = f(x,y), \qquad u_t(x,y,0) = g(x,y). \tag{7.5}$$

As it turns out we can use the formula for three space dimensions for obtaining the formula for two space dimensions. The idea is to regard $u(x,y,t)$ as a function of three variables that don't depend on z. Let us again for simplicity assume that $f \equiv 0$. Since u doesn't depend on z it satisfy the three dimensional wave equation $u_{tt} - u_{xx} - u_{yy} - u_{zz} = 0$ and by the three dimensional formula

$$u(0,0,t) = \frac{1}{4\pi t} \iint_{x^2+y^2+z^2=t^2} g(x,y)\, dS$$

This is twice the integral over the northern hemisphere $z = \sqrt{t^2 - x^2 - y^2}$. Recall that if $z = h(x,y)$ is a surface then $dS = \sqrt{1 + h_x^2 + h_y^2}\, dxdy$. Hence

$$u(0,0,t) = \frac{1}{2\pi t} \iint_{x^2+y^2 \le t^2} g(x,y) \sqrt{1 + h_x^2 + h_y^2}\, dxdy$$

Here

$$1 + h_x^2 + h_y^2 = 1 + \Big(\frac{x}{\sqrt{t^2 - x^2 - y^2}} \Big)^2 + \Big(\frac{y^2}{\sqrt{t^2 - x^2 - y^2}} \Big)^2 = 1 + \frac{x^2 + y^2}{t^2 - x^2 - y^2} = \frac{t^2}{t^2 - x^2 - y^2}$$

so

$$u(0,0,t) = \frac{1}{2\pi} \iint_{x^2+y^2 \le t^2} \frac{g(x,y)}{\sqrt{t^2 - x^2 - y^2}}\, dxdy$$

At another point and when both initial data are non-vanishing we get

$$\begin{aligned} u(x_0,y_0,t_0) = \frac{1}{2\pi} & \iint_{(x-x_0)^2+(y-y_0)^2 \le t_0^2} \frac{g(x,y)}{\sqrt{t_0^2 - (x-x_0)^2 - (y-y_0)^2}}\, dxdy \\ & + \frac{\partial}{\partial t_0} \frac{1}{2\pi} \iint_{(x-x_0)^2+(y-y_0)^2 \le t_0^2} \frac{f(x,y)}{\sqrt{t_0^2 - (x-x_0)^2 - (y-y_0)^2}}\, dxdy \end{aligned} \tag{7.6}$$

Decay of solutions of the wave equation. Suppose that initial data f and g have compact support, i.e. if the vanish outside a compact set, i.e. if some large constant $R \geq 1$:

$$f(\mathbf{x}) = 0, \qquad g(\mathbf{x}) = 0, \qquad \text{when} \qquad |\mathbf{x}| \geq R. \tag{7.7}$$

We also assume that f is continuously differentiable and g is continuous and hence:

$$|f(\mathbf{x})| \leq K, \qquad |f'(\mathbf{x})| \leq K, \qquad \text{and} \qquad |g(\mathbf{x})| \leq K. \tag{7.8}$$

The solution of the wave equation with these data decay:

$$|u(\mathbf{x},t)| \leq \frac{C}{1+t}, \qquad \text{in 3 space dimensions} \tag{7.9}$$

$$|u(\mathbf{x},t)| \leq \frac{C}{\sqrt{1+t}}, \qquad \text{in 2 space dimensions} \tag{7.10}$$

Let us prove (7.9) in case $f \equiv 0$. Then (7.7) and (7.8) can be summarized in

$$|g(\mathbf{x})| \leq KH(R - |\mathbf{x}|). \tag{7.11}$$

where $H(R - |\mathbf{x}|)$ is the step function which is one when $|\mathbf{x}| \leq R$ and 0 when $|\mathbf{x}| > R$.

Let us first prove (7.9). In view of (7.11) we have

$$u(\mathbf{x}_0, t) = \frac{1}{4\pi t_0} \iint_{|\mathbf{x}-\mathbf{x}_0|=t} g(\mathbf{x})\, dS \leq \frac{K}{4\pi t} \iint_{|\mathbf{x}-\mathbf{x}_0|=t} H(R - |\mathbf{x}|)\, dS. \tag{7.12}$$

The last integral is the area of the part of the sphere $|\mathbf{x} - \mathbf{x}_0| = t$, which is inside the ball $|\mathbf{x}| \leq R$. Geometrically, it is clear that this area is bounded by the area of the sphere of radius R, i.e. $4\pi R^2$, and hence that we get the bound $|u(\mathbf{x},t)| \leq KR^2 t^{-1}$.

Let us then go the proof of (7.10). We have

$$u(\mathbf{x}_0, t) = \frac{1}{2\pi} \iint_{|\mathbf{x}-\mathbf{x}_0|\leq t} \frac{g(\mathbf{x})\, d\mathbf{x}}{\sqrt{t^2 - |\mathbf{x}-\mathbf{x}_0|^2}} = \frac{1}{2\pi} \iint_{|\mathbf{z}|\leq t} \frac{g(\mathbf{z}+\mathbf{x}_0)\, d\mathbf{z}}{\sqrt{t^2 - |\mathbf{z}|^2}} \tag{7.12}$$

Using (7.11) and

$$t^2 - |\mathbf{x}-\mathbf{x}_0|^2 = (t + |\mathbf{x}-\mathbf{x}_0|)(t - |\mathbf{x}-\mathbf{x}_0|) \geq t(t - |\mathbf{x}-\mathbf{x}_0|) \tag{7.13}$$

we can bound

$$|u(\mathbf{x}_0, t)| \leq \frac{K}{2\pi\sqrt{t}} \iint_{|\mathbf{z}|\leq t} \frac{H(R - |\mathbf{x}|)\, d\mathbf{x}}{\sqrt{t - |\mathbf{x}-\mathbf{x}_0|}} = \frac{K}{2\pi\sqrt{t}} \iint_{|\mathbf{z}|\leq t} \frac{H(R - |\mathbf{z}+\mathbf{x}_0|)\, d\mathbf{z}}{\sqrt{t - |\mathbf{z}|}} \tag{7.14}$$

If we use (7.8), introduce polar coordinates $\mathbf{z} = (r\cos\theta, r\sin\theta)$ we get

$$|u(\mathbf{x}_0, t)| \leq \frac{K}{2\pi\sqrt{t}} \int_0^t \int_0^{2\pi} \frac{H(R - |(r\cos\theta, r\sin\theta) + \mathbf{x}_0|)\, r\, d\theta\, dr}{\sqrt{t-r}}. \tag{7.15}$$

If we just estimate $H(R - |r\cos\theta, r\sin\theta) + \mathbf{x}_0|) \leq 1$ and integrate we get the estimate $|u(\mathbf{x}_0, t)| \leq 2Kt$. This proves that (7.10) when $t \leq 8R$, say. In what follows we may therefore assume that $t \geq 8R$.

We now divide up into two cases. First, if $|t-|\mathbf{x}_0||\geq 2R$ then $|t-|\mathbf{x}-\mathbf{x}_0||\geq R$, when $|\mathbf{x}|\leq R$ so

$$|u(\mathbf{x}_0,t)|\leq \frac{K}{2\pi\sqrt{t}}\iint_{|\mathbf{x}|\leq R}\frac{d\mathbf{x}}{\sqrt{R}}=\frac{K}{2\pi\sqrt{t}}\frac{\pi R^2}{\sqrt{R}}=\frac{KR^{3/2}}{2\sqrt{t}},\qquad \text{if}\quad |t-|\mathbf{x}_0||\geq 2R. \tag{7.16}$$

We may now assume that $|t-|\mathbf{x}_0||\leq 2R$ and $t\geq 8R$. Let us also assume that $\mathbf{x}_0=(-|\mathbf{x}_0|,0)$ since the integral (7.15) is invariant under rotations. Then $|(r\cos\theta, r\sin\theta)+\mathbf{x}_0|=|(r\cos\theta-|\mathbf{x}_0|, r\sin\theta)|\leq R$, which implies that

$$r\geq|\mathbf{x}_0|-R\geq t-3R\geq 5R,\qquad r|\sin\theta|\leq R. \tag{7.17}$$

Since $|\sin\theta|\leq R/r\leq 1/5$ it follows that $|\theta|\leq 1$ and hence by Taylor's theorem for $\theta>0$, $\sin\theta\geq\theta-\theta^3/6\geq\theta/2$. Therefore we conclude that the integral (7.15) can be estimated by

$$|u(\mathbf{x}_0,t)|\leq\frac{K}{2\pi\sqrt{t}}\int_{t-3R}^{t}\int_{-2R/r}^{2R/r}\frac{r\,d\theta\,dr}{\sqrt{t-r}}=\frac{K}{2\pi\sqrt{t}}\int_{t-3R}^{t}\frac{4R\,dr}{\sqrt{t-r}}=\frac{K}{2\pi\sqrt{t}}4\sqrt{3}R^{3/2}.$$