Lecture 8: 9.3 Rays, Characteristics and Singularities. We are still considering solutions of the wave equation in three space dimension:

$$u_{tt} - c^2 \triangle u = 0, \tag{8.1}$$

where we for simplicity assume that the speed of light $c = 1$. A *light ray* is a path of a point in three space dimensions moving in a straight line at speed c:

$$\mathbf{x} = \mathbf{x}_0 + \mathbf{v}_0\, t, \qquad \text{where} \quad |\mathbf{v}_0| = c. \tag{8.2}$$

The light cone $\{|\mathbf{x} - \mathbf{x}_0| = c|t|\}$ is the union of all light rays originating from the point $\mathbf{x}_0$.

Let S be a three dimensional surface in four dimensional space-time.
Its time slices $S_t = S \cap \{t = \text{const}\}$ are two dimensional surfaces.
S is a *characteristic surface* if it is the union of light rays which are orthogonal to the time slices S_t.
That a light ray (8.1) is orthogonal to S_t is equivalent to that the $\mathbf{v}_0$ is orthogonal to S_t.
Through each point $\mathbf{x}_0$ on S_t there is a whole light cone of rays with $|\mathbf{v}_0| = c = 1$, but only two of them are orthogonal to S_t, one goes in the direction of the outward and the other inward normal.
If $S = \{(\mathbf{x}, t);\ t = \gamma(\mathbf{x})\}$ is a level set there is an analytic description of a characteristic surface:
Theorem The level surface $S = \{t = \gamma(\mathbf{x})\}$ is characteristic if and only if $|\gamma(\mathbf{x})| = 1/c = 1$ for each $\mathbf{x}$.
Proof Suppose first $t = \gamma(\mathbf{x})$ is characteristic. Let $(\mathbf{x}_0, 0)$ be a point on S, i.e. $\gamma(\mathbf{x}_0) = 0$.
By assumption there is light ray $\mathbf{x} = \mathbf{x}_0 + \mathbf{v}_0 t$ which is contained in S so $t = \gamma(\mathbf{x}_0 + \mathbf{v}_0 t)$.
If we take the time derivative and put $t = 0$ we get $1 = \mathbf{v}_0 \cdot \nabla\gamma(\mathbf{x}_0 + \mathbf{v}_0 t) = \mathbf{v}_0 \cdot \nabla\gamma(\mathbf{x}_0)$.
On the other hand the slices $S_0 = \{\mathbf{x};\ \gamma(\mathbf{x}) = \gamma(\mathbf{x}_0)\}$ has $\nabla\gamma(\mathbf{x})$ as its normal.
Since $\mathbf{v}_0$ and $\nabla\gamma(\mathbf{x}_0)$ are normal they must be parallel so $1 = |\mathbf{v}_0 \cdot \nabla\gamma(\mathbf{x}_0)| = |\mathbf{v}_0|\,|\nabla\gamma(\mathbf{x}_0)| = c\,|\nabla\gamma(\mathbf{x}_0)|$.

To prove the converse we assume that $|\nabla\gamma(\mathbf{x})| = 1/c = 1$. Suppose that we have a point $\mathbf{x}_0$ on the initial surface $\gamma(\mathbf{x}_0) = 0$. We will show that the solution of

$$\frac{d\mathbf{x}_i}{dt} = \nabla_i \gamma(\mathbf{x}), \qquad \mathbf{x}(0) = \mathbf{x}_0 \tag{8.3}$$

is in fact alight ray on $t = \gamma(\mathbf{x})$. Taking another time derivative and using the chain rule we get

$$\frac{d^2\mathbf{x}_i}{dt^2} = \sum_{j=1}^{3} \gamma_{ij}(\mathbf{x}) \frac{d\mathbf{x}_j}{dt} = \sum_{j=1}^{3} \gamma_{ij}(\mathbf{x})\gamma_j(\mathbf{x}) \tag{8.4}$$

where $\gamma_i = \partial_i \gamma$, $\gamma_{ij} = \partial_i \partial_j \gamma$. However, since

$$|\nabla\gamma(\mathbf{x})|^2 = \sum_j \gamma_j(\mathbf{x})^2 = 1$$

we get if we apply ∂_i

$$\sum_j 2\gamma_j(\mathbf{x})\gamma_{ij}(\mathbf{x}) = 0,$$

so

$$\frac{d^2\mathbf{x}}{dt^2} = 0.$$

Hence $\dfrac{d\mathbf{x}}{dt} = \nabla\gamma(\mathbf{x}) = \nabla\gamma(\mathbf{x}_0) = \mathbf{v}_0$ is constant along the solutions to (8.3) and hence

$$\mathbf{x} = \mathbf{v}_0 t + \mathbf{x}_0$$

is a light ray. This light ray lies on the level surface since by (8.3)

$$\frac{d}{dt}\gamma(\mathbf{x}) = \nabla\gamma(\mathbf{x}) \cdot \frac{d\mathbf{x}}{dt} = \nabla\gamma(\mathbf{x}) \cdot \nabla\gamma(\mathbf{x}) = 1$$

so

$$\gamma(\mathbf{x}) = t + k.$$

for some constant k. But since $\gamma(\mathbf{x}_0) = 0$ it follows that $k = 1$.

Singularities. Characteristic surfaces are the only surfaces that can carry the singularities of a solution of a wave equation. A singularity is information. A singularity is where the solution or its derivatives are large, in fact, infinite. A singularity is something with a lot of high frequencies.

Let us start with the transport equation:

$$u_t + cu_x = 0, \qquad u(x,0) = f(x).$$

which has the solution

$$u(x,t) = f(x - ct)$$

If the initial data has a singularity $f(x) = H(x)$, where H is the step function, then the singularity will travel exactly along the characteristic line $x - ct = x_0 = 0$. A more elaborate example is:

Ex Suppose that

$$\begin{aligned} u(\mathbf{x},t) &= \frac{1}{2}v(\mathbf{x},t)(t-\gamma(\mathbf{x}))^2, \qquad && \text{for } \gamma(\mathbf{x}) \le t, \\ u(\mathbf{x},t) &= 0, && \text{for } \gamma(\mathbf{x}) \ge t. \end{aligned}$$

where v is a C^2 function (twice continuously differentiable) that is non-vanishing when $t = \gamma(\mathbf{x})$. The second order derivatives will typically have a jump at $t = \gamma(\mathbf{x})$, since e.g. the time derivative is

$$\begin{aligned} u_{tt}(\mathbf{x},t) = \frac{1}{2}v(\mathbf{x})\partial_t^2(t-\gamma(\mathbf{x}))^2 + \frac{1}{2}v_{tt}(\mathbf{x},t)(t-\gamma(\mathbf{x}))^2 + v_t(\mathbf{x},t)\partial_t(t-\gamma(\mathbf{x}))^2 \\ = v(\mathbf{x},t) + (t-\gamma(\mathbf{x})) \times \text{something} = v(\mathbf{x},t), \quad \text{when } t = \gamma(x). \end{aligned}$$

and $u_{tt}(\mathbf{x},t) = 0$, when $t > \gamma(\mathbf{x})$. Similarly the space derivatives are

$$\begin{aligned} \triangle u(\mathbf{x},t) = \frac{1}{2}v(\mathbf{x})\nabla^2(t-\gamma(\mathbf{x}))^2 + \frac{1}{2}\nabla^2 v(\mathbf{x},t)(t-\gamma(\mathbf{x}))^2 + \nabla v(\mathbf{x},t)\cdot\nabla(t-\gamma(\mathbf{x}))^2 \\ = v(\mathbf{x},t)|\nabla\gamma(\mathbf{x})|^2 + (t-\gamma(\mathbf{x})) \times \text{something} = v(\mathbf{x},t)|\nabla\gamma(\mathbf{x})|^2, \quad \text{when } t = \gamma(x). \end{aligned}$$

Hence

$$u_{tt}(\mathbf{x},t) - \triangle u(\mathbf{x},t) = v(\mathbf{x},t)(1 - |\gamma(\mathbf{x})|^2), \qquad \text{when } t = \gamma(\mathbf{x}).$$

Since $u_{tt}(\mathbf{x},t) - \triangle u(\mathbf{x},t) = 0$, when $t > \gamma(\mathbf{x})$ it follows that $u_{tt}(\mathbf{x},t) - \triangle u(\mathbf{x},t)$ does not have a jump at $t = \gamma(\mathbf{x})$ if and only if

$$|\gamma(\mathbf{x})|^2 = 1$$

i.e. if the level surface $t = \gamma(\mathbf{x})$ is characteristic.