Lecture 9: 9.1 Invariance of the wave operator under Lorentz transformations and 9.3 The wave equation with a Source.

6.1 Invariance of the Laplacian under rotations. A rotation or orthogonal transformation $\mathbf{R}^n \ni \mathbf{x} \to Q\mathbf{x} \in \mathbf{R}^n$, is multiplication by an orthogonal matrix $Q = (q_{ij})$.
An orthogonal matrix satisfies $Q^T Q = I$ or equivalently $\langle Q\mathbf{x}, Q\mathbf{y}\rangle = \langle \mathbf{x}, \mathbf{y}\rangle$ for any $\mathbf{x}, \mathbf{y} \in \mathbf{R}^n$.

For an arbitrary $n\times n$ matrix $A = (a_{ij})$ the kth component of the vector $A\mathbf{x}$ is $(A\mathbf{x})_k = \sum_\ell a_{k\ell}x_\ell$. By the change rule we have

$$\partial_i u(A\mathbf{x}) = \sum_{k=1}^n (\partial_k u)(A\mathbf{x})\, \partial_i (A\mathbf{x})_k = \sum_{k=1}^n (\partial_k u)(A\mathbf{x})\, a_{ki}$$

and hence

$$\partial_i\partial_j u(A\mathbf{x}) = \sum_{k=1}^n \sum_{\ell=1}^n (\partial_k\partial_\ell u)(A\mathbf{x})\, a_{ki}a_{\ell j} \tag{9.1}$$

If we now apply this to an orthogonal matrix A and sum over $i = j$ we get

$$\triangle u(Q\mathbf{x}) = \sum_{k=1}^n \sum_{\ell=1}^n (\partial_k\partial_\ell u)(Q\mathbf{x}) \sum_{i=1}^n q_{ki}q_{\ell i}$$

If $Q_{ij} = q_{ij}$ then $Q^T_{ij} = q_{ji}$. Hence $(QQ^T)_{k\ell} = \sum_{i=1}^n Q_{ki}Q_{i\ell} = \sum_{i=1}^n q_{ki}q_{\ell i}$. But $(QQ^T)_{k\ell} = I_{k\ell}$ so

$$\triangle\big(u(Q\mathbf{x})\big) = \sum_{k=1}^n \sum_{\ell=1}^n (\partial_k\partial_\ell u)(Q\mathbf{x})\, I_{k\ell} = (\triangle u)(Q\mathbf{x}).$$

In two dimensions the rotation matrices are $Q = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

9.1 Invariance of the wave operator under Lorentz transformations. A Lorentz transformation $\mathbf{x} \to L\mathbf{x}$ is multiplication with a matrix $L = (\ell_{ij})$ such that $L^T\Gamma L = \Gamma$, where $\Gamma = \mathrm{diag}\{1, \dots, 1, -1\}$, i.e. Γ is the diagonal matrix with one's in the diagonal apart from a negative one in the lower right corner. Equivalently $\langle \Gamma L\mathbf{x}, L\mathbf{x}\rangle = \langle \Gamma\mathbf{x}, \mathbf{x}\rangle = x_1^2 + \cdots + x_n^2 - x_{n+1}^2$.
If we apply (9.1) to $A = L$, multiply (9.1) by Γ_{ij} and sum over i, j we get

$$\Box u(L\mathbf{x}) = \sum_{i=1}^{n+1}\sum_{j=1}^{n+1} \Gamma_{ij}\partial_i\partial_j u(L\mathbf{x}) = \sum_{k=1}^{n+1}\sum_{m=1}^{n+1} (\partial_k\partial_m u)(L\mathbf{x}) \sum_{i=1}^{n+1}\sum_{j=1}^{n+1} \ell_{ki}\ell_{mj}\Gamma_{ij}$$

But $\sum_{i=1}^{n+1}\sum_{j=1}^{n+1} \ell_{ki}\ell_{mj}\Gamma_{ij} = \sum_{i=1}^{n+1}\sum_{j=1}^{n+1} L_{ki}\Gamma_{ij}L^T_{jm} = (L\Gamma L^T)_{km} = \Gamma_{km}$ so we get

$$\Box\big(u(L\mathbf{x})\big) = \sum_{k=1}^{n+1}\sum_{m=1}^{n+1} (\partial_k\partial_m u)(Q\mathbf{x})\, \Gamma_{mk} = (\Box u)(L\mathbf{x}).$$

In $\mathbf{R}^{1+1}$ the Lorentz transformations are given by

$$(x,t) \to L(x,t) = \Big(\frac{x-\beta t}{\sqrt{1-\beta^2}}, \frac{t-\beta x}{\sqrt{1-\beta^2}}\Big), \qquad L = \frac{1}{\sqrt{1-\beta^2}}\begin{bmatrix} 1 & -\beta \\ -\beta & 1 \end{bmatrix}.$$

There are different ways of writing this. It is simply a linear transformation such that

$$\text{if} \qquad (y,s) = L(x,t), \qquad \text{then} \qquad y^2 - s^2 = x^2 - t^2.$$

9.3 The wave equation with a Source. Recall the formula for the solution of the initial value problem for the wave equation in 3 space dimensions:

$$u_{tt} - \triangle u = 0, \qquad u(\mathbf{x},0) = f(\mathbf{x}), \qquad u_t(\mathbf{x},0) = g(\mathbf{x}). \tag{9.2}$$

We have

$$u(\mathbf{x},t) = \mathcal{S}g(\mathbf{x},t) + \frac{\partial}{\partial t}\mathcal{S}f(\mathbf{x},t),$$

where

$$\mathcal{S}g(\mathbf{x},t) \equiv \frac{1}{4\pi t}\iint_{|\mathbf{z}-\mathbf{x}|=t} g(\mathbf{z})\,dS(\mathbf{z}) = \frac{1}{4\pi t}\iint_{|\mathbf{y}|=t} g(\mathbf{x}+\mathbf{y})\,dS(\mathbf{y}), \tag{9.3}$$

where $dS(\mathbf{y})$ is the surface measure on the sphere of radius t centered at $\mathbf{x}$. Note that the sphere $|\mathbf{z}-\mathbf{x}| = t$ is the intersection of the backward light cone from $(\mathbf{x},t)$ with the initial surface $t = 0$.

By *Duhamel's principle* the solution to

$$u_{tt}(\mathbf{x},t) - \triangle u(\mathbf{x},t) = F(\mathbf{x},t), \qquad u(\mathbf{x},0) = 0, \qquad u_t(\mathbf{x},0) = 0. \tag{9.4}$$

is given by

$$u(\mathbf{x},t) = \int_0^t \mathcal{S}F(\cdot,s)(\mathbf{x},t-s)\,ds = \int_0^t \frac{1}{4\pi(t-s)}\iint_{|\mathbf{y}|=t-s} F(\mathbf{x}+\mathbf{y},s)\,dS(\mathbf{y})\,ds, \tag{9.5}$$

where $\mathcal{S}F(\cdot,s)$ acts on the function of the first variables only and s is a fixed parameter.(section 3.4) The proof of this used that by definition $v(\mathbf{x},t;s) = \mathcal{S}F(\cdot,s)(\mathbf{x},t-s)$ satisfy the initial value problem

$$v_{tt} - \triangle v = 0, \qquad v(\mathbf{x},s;s) = 0, \quad v_t(\mathbf{x},s;s) = F(\mathbf{x},s).$$

Hence

$$\frac{\partial}{\partial t}u(\mathbf{x},t) = \frac{\partial}{\partial t}\int_0^t v(\mathbf{x},t;s)\,ds = v(\mathbf{x},t;t) + \int_0^t v_t(\mathbf{x},t;s)\,ds = \int_0^t v_t(\mathbf{x},t;s)\,ds$$

whereas

$$\frac{\partial^2}{\partial t^2}u(\mathbf{x},t) = \frac{\partial}{\partial t}\int_0^t v_t(\mathbf{x},t;s)\,ds = v_t(\mathbf{x},t;t) + \int_0^t v_{tt}(\mathbf{x},t;s)\,ds = F(\mathbf{x},t) + \int_0^t v_{tt}(\mathbf{x},t;s)\,ds$$

and it follows that

$$(\partial_t^2 - \triangle)u = F(\mathbf{x},t) + \int_0^t (v_{tt} - \triangle v)(\mathbf{x},t;s)\,ds = 0.$$

We can write (9.5)

$$u(\mathbf{x},t) = \frac{1}{4\pi}\int_0^t \iint_{|\mathbf{y}|=t-s} \frac{F(\mathbf{x}+\mathbf{y},t-|\mathbf{y}|)}{|\mathbf{y}|}\,dS(\mathbf{y})\;ds. \tag{9.6}$$

This is an iterated integral in spherical coordinates and $d\mathbf{y} = dS(\mathbf{y})dr = dS(\mathbf{y})ds$, where $r = t-s = |\mathbf{y}|$. The domain of integration is the backward light cone from $(\mathbf{x},t)$. Hence we can also write:

$$u(\mathbf{x},t) = \frac{1}{4\pi}\iiint_{|\mathbf{y}|\le t} \frac{F(\mathbf{x}+\mathbf{y},t-|\mathbf{y}|)}{|\mathbf{y}|}\,d\mathbf{y} = \frac{1}{4\pi}\iiint_{|\mathbf{x}-\mathbf{z}|\le t} \frac{F(\mathbf{z},t-|\mathbf{x}-\mathbf{z}|)}{|\mathbf{x}-\mathbf{z}|}\,d\mathbf{z}. \tag{9.7}$$