

Math 132A PDE Practice Midterm I, Spring 08, Lindblad.

1. (a) Solve the initial value problem for the transport equation:

$$2u_t + 3u_x = 0, \quad u(x, 0) = \sin x.$$

(b) Solve the initial value problem for the inhomogeneous wave equation:

$$u_{tt} - u_{xx} = e^{ax}, \quad u(x, 0) = 0, \quad u_t(x, 0) = 0.$$

2) Solve the initial value problem for the diffusion equation in three space dimensions:

$$u_t - \Delta u = 0, \quad u(x, y, z, 0) = xy^2z.$$

3). Suppose that u satisfies the Klein-Gordon equation in three space dimensions:

$$u_{tt} - \Delta u + u = 0.$$

and that $u(\mathbf{x}, t) = 0$ when $|\mathbf{x}|$ is large. Prove that the energy

$$E(t) = \frac{1}{2} \iiint u_t(\mathbf{x}, t)^2 + |\nabla u(\mathbf{x}, t)|^2 d\mathbf{x} + \frac{1}{2} \iiint u(\mathbf{x}, t)^2 d\mathbf{x},$$

is conserved, i.e. independent of t .

4) Suppose that u satisfies Laplace equation in two space dimensions:

$$\Delta u = 0.$$

Let $Q = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$, where $a = \cos \theta$ and $b = \sin \theta$, be a rotation matrix.

Set $v(\mathbf{x}) = u(Q\mathbf{x}) = u(ax - by, bx + ay)$. Show that

$$\Delta v = 0.$$

5) (a) Find the solution to the wave equation in three space dimensions:

$$u_{tt} - \Delta u = 0, \quad u(\mathbf{x}, 0) = 0, \quad u_t(\mathbf{x}, 0) = e^{-|\mathbf{x}|^2},$$

(b) Let u be the solution in (a) and let ω be a unit vector in \mathbf{R}^3 . Show that the limit

$$F(\rho, \omega) = \lim_{t \rightarrow \infty} t \cdot u((t + \rho)\omega, t)$$

exists.