Lecture 11: Section 5.3: Exponentials of operators. Just like the exponential function was used to solve an ordinary differential equation we will see that one can take the exponential of operators and matrices and this will be useful for solving systems of differential equations. The exponential of an operator is defined by suing the power series for the exponential function:

$$e^T = \lim_{N\to\infty} \sum_{k=0}^{N} \frac{T^k}{k!} \tag{5.3.1}$$

Each term in the series is well defined if T is an operator on $\mathbf{R}^n$ or equivalently a matrix, but we have to give a meaning to convergence of the infinite sum. In a basis we could mean that the elements of the matrix defined by finite sums of the form (5.3.1) converges as $N \to \infty$. Recall that a sum of real numbers $\lim_{N\to\infty} \sum_{k=0}^{N} \alpha_k$ converges if it can be *dominated* $|\alpha_k| \le c_k$ by a series that *converges* $\sum_{k=0}^{\infty} c_k \le C < \infty$. For matrices, one can use many different norms that are all comparable in size. In general a norm is a measure of size such that

$$\|x\| \ge 0, \quad \|x\| = 0 \Leftrightarrow x = 0, \qquad \|x+y\| \le \|x\| + \|y\|, \qquad \|\alpha x\| = |\alpha| \|x\| \tag{5.3.2}$$

It is however best to use the so called operator norm

$$\|T\| = \sup_{|x|\le 1} |Tx| = \sup_{x\ne 0} \frac{|Tx|}{|x|} \tag{5.3.3}$$

where $|x|$ and $|Tx|$ are the usual norms of vectors in $\mathbf{R}^n$. If $A = [a_{ij}]$ is the matrix for T in the standard basis then obviously $\max_{i,j} |a_{ij}| \le \|T\|$, which is seen by picking $x = e_j$ in (5.3.3). We shall now prove that (5.3.1) converges. First

$$\|ST\| \le \|S\| \, \|T\| \tag{5.3.4}$$

In fact $|STx| \le \|S\| \, |Tx| \le \|S\| \, \|T\| \, |x|$, for all x which implies (5.3.4). Hence $\|T^k/k!\| \le \|T\|^k/k!$ and $\sum_{k=0}^{\infty} \|T\|^k/k! = e^{\|T\|} < \infty$ so it follows that the elements of (5.3.1) are sums of real numbers that are bounded by a sequence $\|T\|^k/k!$ whose sum converges. Therefore the matrices (5.3.1) converges.

Proposition. *We have:*
(a) if $Q = PTP^{-1}$*, then* $e^Q = Pe^TP^{-1}$*;*
(b) if $ST = TS$*; then* $e^{S+T} = e^S e^T$*;*
(c) $e^{-S} = (e^S)^{-1}$*;*
(d) if $T_{a,b} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$*; then* $e^{T_{a,b}} = e^a \begin{bmatrix} \cos b & -\sin b \\ \sin b & \cos b \end{bmatrix}$*;*

Proof. The proof of (a) follows from that $P\big(\sum_{k=1}^{N} T^k/k!\big)P^{-1} = \sum_{k=1}^{N} PT^kP^{-1}/k! = \sum_{k=1}^{N} PTP^{-1}PTP^{-1} \cdots PTP^{-1}/k! = \sum_{k=1}^{N} (PTP^{-1})^k/k!$, by taking the limit as $N \to \infty$. To prove (b) we first note that, if $ST = TS$ then by the binomial theorem $(S+T)^n = n! \sum_{j+k=n} (S^j/j!)(T^k/k!)$. Formally changing the order of summation: $e^{S+T} = \sum_{n=0}^{\infty} (S+T)^n/n! = \sum_{n=0}^{\infty} \sum_{j+k=n} (S^j/j!)(T^k/k!) = \big(\sum_{j=0}^{\infty} (S^j/j!)\big) \sum_{k=0}^{\infty} (T^k/k!)\big) = e^S e^T$. However, because the sums are infinite it requires some justification, see the book. (c) is the special case of (b) with $T = -S$.

The proof of (d) follows from the correspondence $T_{a,b} = \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \leftrightarrow a + ib$, from chapter 3, i.e. applying the operator $T_{a,b}$ to vectors $(x, y) \in \mathbf{R}^2$ corresponds to multiplying the corresponding complex number $x + iy \in \mathbf{C}$ by $a + ib$. This preserves sums, products, real multiples and limits, i.e. a sum of operators corresponds to the sum of the corresponding numbers etc. Therefore, taking the power series that defines the exponential function for the operator corresponds to forming the same the series for the exponential function of a complex number $e^{T_{a,b}} \leftrightarrow e^a e^{ib}$, where $e^{ib} = \sum_{k=0}^{\infty} \dfrac{(ib)^k}{k!} = \sum_{k=0}^{\infty} \dfrac{(-1)^k b^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \dfrac{(-1)^k b^{2k+1}}{(2k+1)!} = \cos b + i \sin b$. $\square$

Ex. Let $T = \begin{bmatrix} a & 0 \\ b & a \end{bmatrix}$. We write $T = aI + B$, where $B = \begin{bmatrix} 0 & 0 \\ b & 0 \end{bmatrix}$. Since any operator commutes with a multiple of the identity we have $e^T = e^{aI} e^B = e^a e^B$. Now $B^2 = 0$ so $e^B = \sum_{k=0}^{\infty} B^k \frac{1}{k!} = I + B$ and hence $e^T = e^a \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} = \begin{bmatrix} e^a & 0 \\ e^a b & e^a \end{bmatrix}$.

Ex. If $T = \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}$ then $e^T = \begin{bmatrix} e^\lambda & 0 \\ 0 & e^\mu \end{bmatrix}$, which follows from using the Taylor series since since $\begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix}^k = \begin{bmatrix} \lambda^k & 0 \\ 0 & \mu^k \end{bmatrix}$.

Ex. If $T = P \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} P^{-1}$ then $e^T = P \begin{bmatrix} e^\lambda & 0 \\ 0 & e^\mu \end{bmatrix} P^{-1}$.

Ex. If $T = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}$ then by an example above $e^T = e^\lambda \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.

An alternative approach to the power series of the operator is to note that each eigenvector for T with a real eigenvalue λ is also an eigenvector for e^T with an eigenvalue e^λ. This follows from using the Taylor series definition. (5.3.1). On the other if we have distinct eigenvalues and hence a basis of eigenvectors this completely defines e^T on any vector.