Lecture 12: 4.3 Revisited.

Ex 2 Find the solution to the system: $\quad x' = Ax, \qquad A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & -2 \\ 0 & 1 & 0 \end{bmatrix}$

The eigenvalues are $\lambda_1 = 1$ and $\lambda_2 = i\sqrt{2}$, $\lambda_3 = \overline{\lambda_2} = -i\sqrt{2}$.
(Complex) Eigenvectors are found by solving $(A - \lambda_i I)\hat{e}_i = 0$, for $i = 1, 2, 3$.

$$i = 1: \quad \begin{bmatrix} 0 & 0 & 1 \\ 0 & -1 & -2 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ e.g. } \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$i = 2: \quad \begin{bmatrix} 1 - i\sqrt{2} & 0 & 1 \\ 0 & -i\sqrt{2} & -2 \\ 0 & 1 & -i\sqrt{2} \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \text{ e.g. } \begin{bmatrix} w_1 \\ w_2 \\ w_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 - i\sqrt{2} \\ -1 + i\sqrt{2} \end{bmatrix}.$$

Since the matrix is real the complex conjugate of an eigenvector is also an eigenvector so we get the complex eigenvectors $\hat{e}_1 = (1, 0, 0)$, $\hat{e}_2 = (1, -2 - i\sqrt{2}, -1 + i\sqrt{2})$ and $\hat{e}_3 = (1, -2 + i\sqrt{2}, -1 - i\sqrt{2})$.

There are now a couple of different ways to proceed. One is to make a complex change of variables transforming the matrix to a complex diagonal matrix and solving the complex system of differential equations. The other way is to making a real change of variables transforming the matrix to a standard form, which is not quite diagonal but also has a 2×2 block corresponding to a rotation matrix in the plane and using that we already know how to solve the resulting standard system.

Complex diagonalization With the complex change of variables $x = \hat{P}\hat{x}$ where

$$\hat{P} = \begin{bmatrix} \hat{e}_1 \hat{e}_2 \hat{e}_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 - i\sqrt{2} & -2 + i\sqrt{2} \\ 0 & -1 + i\sqrt{2} & -1 - i\sqrt{2} \end{bmatrix},$$

we get the system

$$\hat{x}' = \hat{A}\hat{x}, \qquad \hat{A} = \hat{P}^{-1} A \hat{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & i\sqrt{2} & 0 \\ 0 & 0 & -i\sqrt{2} \end{bmatrix}$$

which has the solution

$$\begin{bmatrix} \hat{x}_1 \\ \hat{x}_2 \\ \hat{x}_3 \end{bmatrix} = \begin{bmatrix} \hat{a}_1 e^{t} \\ \hat{a}_2 e^{i\sqrt{2}t} \\ \hat{a}_3 e^{-i\sqrt{2}t} \end{bmatrix}$$

Transforming back gives

$$x = \hat{P}\hat{x} = \hat{a}_1 e^{t} \hat{e}_1 + \hat{a}_2 e^{i\sqrt{2}t} \hat{e}_2 + \hat{a}_3 e^{-i\sqrt{2}t} \hat{e}_3$$

for some complex constants $\hat{a}_i$, $i = 1, 2, 3$. We remark that, since the matrix is real it follows that the solution is real for all times if it is real when $t = 0$. (This follows because the complex conjugate of the solution is also a solution so the imaginary part is a solution that vanishes when $t = 0$ and hence by uniqueness vanish for all t.

The real standard form: We will now choose a different, real, basis $\tilde{e}_1 = \hat{e}_1$ but $\tilde{e}_2$ and $\tilde{e}_3$ will not be eigenvectors but instead the real and imaginary part of the complex eigenvectors: $\hat{e}_2 = \tilde{e}_3 + i\,\tilde{e}_2$, where $\tilde{e}_3 = (1, -2, -1)$ and $\tilde{e}_2 = (0, -\sqrt{2}, \sqrt{2})$.

With the real change of variables $x = \tilde{P}\tilde{x}$ where

$$\tilde{P} = \begin{bmatrix} \tilde{e}_1 \tilde{e}_2 \tilde{e}_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -\sqrt{2} & -2 \\ 0 & \sqrt{2} & -1 \end{bmatrix},$$

we get the system

$$\tilde{x}' = \tilde{A}\tilde{x}, \qquad \tilde{A} = \tilde{P}^{-1} A \tilde{P} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -\sqrt{2} \\ 0 & \sqrt{2} & 0 \end{bmatrix}$$

Which reduces it to solving the two standard systems:

$$\tilde{x}_1' = \tilde{x}_1, \qquad \begin{bmatrix} \tilde{x}_2' \\ \tilde{x}_3' \end{bmatrix} = \begin{bmatrix} 0 & -\sqrt{2} \\ \sqrt{2} & 0 \end{bmatrix} \begin{bmatrix} \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix}$$

The later is the system studied in section 3.4. It corresponds to a rotation in the $(\tilde{x}_2, \tilde{x}_3)$ plane of the initial vector $(\tilde{a}_2, \tilde{a}_3)$ with an angular velocity of $\sqrt{2}$:

$$\begin{bmatrix} \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix} = \begin{bmatrix} \cos\sqrt{2}t & -\sin\sqrt{2}t \\ \sin\sqrt{2}t & \cos\sqrt{2}t \end{bmatrix} \begin{bmatrix} \tilde{a}_2 \\ \tilde{a}_3 \end{bmatrix}.$$

We hence get

$$\begin{bmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \tilde{x}_3 \end{bmatrix} = \begin{bmatrix} \tilde{a}_1 e^t \\ \tilde{a}_2 \cos\sqrt{2}t - \tilde{a}_3 \sin\sqrt{2}t \\ \tilde{a}_3 \sin\sqrt{2}t + \tilde{a}_2 \cos\sqrt{2}t \end{bmatrix}$$

and

$$x = \tilde{P}\tilde{x} = \tilde{a}_1 e^t\, \tilde{e}_1 + \big(\tilde{a}_2 \cos\sqrt{2}t - \tilde{a}_3 \sin\sqrt{2}t\big)\tilde{e}_2 + \big(\tilde{a}_3 \sin\sqrt{2}t + \tilde{a}_2 \cos\sqrt{2}t\big)\tilde{e}_3$$

There are several choices of basis vectors $\tilde{e}_2$ and $\tilde{e}_3$ corresponding to different choices of complex eigenvector $\hat{e}_2 = (w_1, w_2, w_3)$. In fact we can multiply $\hat{e}_2$ be any complex constant $\alpha + i\beta$ and it is still an eigenvector but with a different decomposition into real and imaginary parts. If e.g. we multiply with $(1+i\sqrt{2})$ we get $(1+i\sqrt{2})\hat{e}_2 = (1+i\sqrt{2})(1, -2-i\sqrt{2}, -1+i\sqrt{2}) = (1+i\sqrt{2}, -i3\sqrt{2}, -3) = \check{e}_3 + i\check{e}_2$, where $\check{e}_3 = (1, 0, -3)$ and $\check{e}_2 = (\sqrt{2}, -3\sqrt{2}, 0)$. Let $\check{e}_1 = \hat{e}_1$. We can then form $\check{P} = [\check{e}_1\, \check{e}_2\, \check{e}_3]$ and make the change of variable $x = \check{P}\check{x}$ in which the system takes the form $\check{x}' = \check{A}\check{x}$, where $\check{A} = \check{P}A\check{P}^{-1} = \ldots = \tilde{A}$. The solution $\check{x}$ therefore look exactly the same as $\tilde{x}$ but with some different constants $(\check{a}_1, \check{a}_2, \check{a}_3)$ replacing $(\tilde{a}_1, \tilde{a}_2, \tilde{a}_3)$ so we get

$$x = \check{P}\check{x} = \check{a}_1 e^t\, \check{e}_1 + \big(\check{a}_2 \cos\sqrt{2}t - \check{a}_3 \sin\sqrt{2}t\big)\check{e}_2 + \big(\check{a}_3 \sin\sqrt{2}t + \check{a}_2 \cos\sqrt{2}t\big)\check{e}_3$$

Since $\{\check{e}_2, \check{e}_3\}$ is different from $\{\tilde{e}_2, \tilde{e}_3\}$ this appears to be a different answer. However, for each choice of $(\tilde{a}_2, \tilde{a}_3)$ there is a corresponding choice of $(\check{a}_2, \check{a}_3)$ so that the two expressions are equal. The transformation is a multiplication and a rotation:

$$\begin{bmatrix} \check{a}_2 \\ \check{a}_3 \end{bmatrix} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1/\sqrt{3} & -\sqrt{2}/\sqrt{3} \\ \sqrt{2}/\sqrt{3} & 1/\sqrt{3} \end{bmatrix} \begin{bmatrix} \tilde{a}_2 \\ \tilde{a}_3 \end{bmatrix}.$$

corresponding to multiplying by $1 + i\sqrt{2}$ in the complex plane:
$\check{a}_1 + i\check{a}_2 = (1 + i\sqrt{2})(\tilde{a}_1 + i\tilde{a}_2)$.