Lecture 14: Chapter 6: The standard form of operators. If $A : E \to E$ is an operator on $E = \mathbf{R}^n$ we want to decompose the vector space into a *direct sum* $E = E_1 \oplus \cdots \oplus E_r$, i.e. any $x \in E$ can be written in a unique way as $x = x_1 + ... + x_r$ with $x_i \in E_i$, of subspaces that are *invariant* under A, $A : E_i \to E_i$. Furthermore we want $A_i = A\big|_{E_i}$ (*A restricted to E_i*) to have a simple standard form. In particular $A_i = S_i + N_i$, where S_i is *semisimple*, i.e. complex diagonalizable, and N_i is *nilpotent*, i.e. $N_i^k = 0$ for some k. From Chapter 5 we know that it is simple to calculate the exponential matrix for such a sum and we can hence solve $x' = Ax$.

Section 6.1: The primary decomposition. Let $T : E \to E$ be an operator on a vector space E. Throughout this section we assume that either E is complex or all eigenvalues of T are real and E is real. Let the characteristic polynomial for T be

$$p(t) = (t - \lambda_1)^{n_1} \cdots (t - \lambda_r)^{n_r}, \tag{6.1.1}$$

where $n_1 + ... + n_r = \dim E$. Let E_k be the *generalized eigenspace* of λ_k:

$$E_k = \operatorname{Ker}(T - \lambda_k)^{n_k} \tag{6.1.2}$$

which is as an invariant subspace under T, i.e. $TE_k \subset E_k$. We claim that:

Theorem. *We have $E = E_1 \oplus \cdots \oplus E_r$ and* $\dim E_k = n_k$

If all $n_i = 1$ it follows from diagonalization and we will just prove it when $r = 1$. Note that $\operatorname{Ker} T^j \subset \operatorname{Ker} T^{j+1}$ and $\operatorname{Im} T^{k+1} \subset \operatorname{Im} T^k$, and since the vector spaces are finite dimensional $\operatorname{Ker} T^j = \operatorname{Ker} T^{j+1}$ and $\operatorname{Im} T^{k+1} = \operatorname{Im} T^k$ for large j and k. Let n and m be the smallest integers such that

$$N = \operatorname{Ker} T^n = \operatorname{Ker} T^j, \quad j \geq n, \qquad M = \operatorname{Im} T^m = \operatorname{Im} T^k, \quad k \geq m \tag{6.1.3}$$

Lemma 1. *$E = N \oplus M$, $T : M \to M$, $T : N \to N$ and $T\big|_M$ is invertible.*

Proof. It follows directly from (6.1.3) that $TN \subset N$ and $TM = M$ so $T\big|_M$ is invertible. Let $S = T^p$, where $p = \max(m, n)$. Then $\operatorname{Ker} S = N$, $\operatorname{Im} S = M$ and $SM = M$ so $S\big|_M$ is invertible. If $x \in E$, then $Sx \in M$ so there is $z \in M$ such that $Sz = Sx$ and hence $S(z - x) = 0$ and we can write $x = (x - z) + z$ where $x - z \in N$ and $z \in M$. Furthermore, this decomposition is unique since $N \cap M = \{0\}$. □

Let us now prove the theorem in case there is only one eigenvalue that we may assume to be 0. Then by Lemma 1, $T\big|_M$ has no eigenvalues which is impossible unless $M = \{0\}$, and therefore $E + N$. We need to show that $E_1 = N$, i.e. that $n \leq n_1 = \dim E$, but this follows from Lemma 2 below:

Lemma 2. *Suppose that $T^n x = 0$ and $T^{n-1} x \neq 0$ then the vectors $T^j x$, for $0 \leq j \leq n - 1$ are independent.*

Proof. Suppose that $T^j x$ are linearly dependent, i.e. $\sum_{j=0}^{n-1} a_j T^j x = 0$ and let k be the smallest integer such that $a_k \neq 0$. Then applying T^{n-k-1} to the sum gives since $T^n x = 0$ that $a_k T^{n-1} x = 0$ so $T^{n-1} x = 0$ which contradicts our assumption. □